Weakly Projective MV-algebras

Leonardo Manuel Cabrer

University of Bern

Outline

- Introduction and motivation
- Combinatorial machinery
- Characterization of Weakly Projective MV-algebras
- Connections with admissibility

Theorem (McNaughton (1951))

For each $n = 1, 2, \ldots$, the MV-algebra $\mathcal{M}([0,1]^n)$ of McNaughton maps from the n-cube is freely generated by the projection maps $\xi_i(a_1, \ldots, a_n) = a_i$.

A **McNaughton map** is a continuous function $f : [0,1]^n \to [0,1]$ satisfying:

There are linear polynomials $p_1, \ldots, p_m$ with integer coefficients, such that for all $x \in [0,1]^n$ there is $i \in \{1, \ldots, m\}$ with $f(x) = p_i(x)$.

Introduction

Finitely Presented MV-algebras and Rational Polyhedra

An MV-algebra A is **finitely presented** if there exist n and $f \in \mathcal{M}([0,1]^n)$ such that

$$A \cong \mathcal{M}([0,1]^n) / \mathrm{cong}(\mathrm{f}, 1)$$

Introduction

Finitely Presented MV-algebras and Rational Polyhedra

An MV-algebra A is **finitely presented** if there exist n and $f \in \mathcal{M}([0,1]^n)$ such that

$$A \cong \mathcal{M}([0,1]^n) \,/\mathrm{cong}(\mathrm{f}, 1) \cong \mathcal{M}([0,1]^n) \restriction f^{-1}(1)$$

Introduction

Finitely Presented MV-algebras and Rational Polyhedra

An MV-algebra A is **finitely presented** if there exist n and $f \in \mathcal{M}([0,1]^n)$ such that

$$A \cong \mathcal{M}([0,1]^n)\,/\mathrm{cong}(\mathrm{f},1) \cong \mathcal{M}(f^{-1}(1))$$

Introduction

Finitely Presented MV-algebras and Rational Polyhedra

A **rational polyhedron** P in $[0,1]^n$ is a finite union of closed simplexes $P = S_1 \cup \cdots \cup S_t$ in $[0,1]^n$ such that the coordinates of the vertices of every simplex S_i are rational numbers.

Introduction

Finitely Presented MV-algebras and Rational Polyhedra

A **rational polyhedron** P in $[0,1]^n$ is a finite union of closed simplexes $P = S_1 \cup \cdots \cup S_t$ in $[0,1]^n$ such that the coordinates of the vertices of every simplex S_i are rational numbers.

Theorem

A subset $P \subseteq [0,1]^n$ is a rational polyhedron if and only if it there exists $f \in \mathcal{M}([0,1]^n)$ such that $P = f^{-1}(1)$.

Introduction

Finitely Presented MV-algebras and Rational Polyhedra

An MV-algebra A is finitely presented if there exist n and a rational polyhedron $P \subseteq [0,1]^n$ such that

$$A \cong \mathcal{M}(P)$$

Introduction

Homomorphisms and $\mathbb{Z}$-maps

$$\mathcal{M}(P \subseteq [0,1]^n) \xrightarrow{h} \mathcal{M}(Q \subseteq [0,1]^m)$$

$$\xi_i \longmapsto h(\xi_i)\colon Q \to [0,1]$$

Introduction

Homomorphisms and $\mathbb{Z}$-maps

$$\mathcal{M}(P \subseteq [0,1]^n) \xrightarrow{\ h\ } \mathcal{M}(Q \subseteq [0,1]^m)$$

$$\xi_i \longmapsto h(\xi_i)\colon Q \to [0,1]$$

$$\eta_h = (h(\xi_1), \ldots, h(\xi_n))\colon Q \to [0,1]^n$$

$$P \xleftarrow{\ \eta_h\ } Q$$

Introduction

Homomorphisms and $\mathbb{Z}$-maps

$$\mathcal{M}(P \subseteq [0,1]^n) \xrightarrow{h} \mathcal{M}(Q \subseteq [0,1]^m)$$

$$\xi_i \longmapsto h(\xi_i)\colon Q \to [0,1]$$

$$\eta_h = (h(\xi_1), \ldots, h(\xi_n))\colon Q \to [0,1]^n$$

$$P \xleftarrow{\eta_h} Q$$

$$h(f) = f \circ \eta_h$$

Definition

Given rational polyhedra $P \subseteq [0,1]^n$ and $Q \subseteq [0,1]^m$ a continuous map $\eta\colon P \to Q$ is called a **$\mathbb{Z}$-map** if there are finite affine linear maps $q_1, \ldots q_k$ such that for each $x \in P$, $\eta(x) = q_i(x)$ for some $i = 1, \ldots, k$.

Introduction

Homomorphisms and $\mathbb{Z}$-maps

Definition

Given rational polyhedra $P \subseteq [0,1]^n$ and $Q \subseteq [0,1]^m$ a continuous map $\eta \colon P \to Q$ is called a **$\mathbb{Z}$-map** if there are finite affine linear maps $q_1, \ldots q_k$ such that for each $x \in P$, $\eta(x) = q_i(x)$ for some $i = 1, \ldots, k$.

Lemma

η is a $\mathbb{Z}$-map if and only if $\xi_i \circ \eta \in \mathcal{M}(P)$ for each $i = 1, \ldots, m$.

Introduction

Duality

The category of Rational Polyhedra with $\mathbb{Z}$-maps is dually equivalent to the category of finitely presented MV-algebras.

- Projectives are retractions of Free algebras.
- Retractions are preserved under dualities.

- Projectives are retractions of Free algebras.
- Retractions are preserved under dualities.

Theorem

A finitely generated MV-algebra is projective iff there exist a number $n = 1, 2, \ldots$ and a $\mathbb{Z}$-map $\eta\colon [0,1]^n \to [0,1]^n$ such that

(i) $\eta \circ \eta = \eta$,

(ii) $A \cong \mathcal{M}(\eta([0,1]^n))$.

[LMC & D. Mundici, Projective MV-algebras and rational polyhedra, *Algebra Universalis* Volume 62, Number 1, 63-74]

Introduction

Duality: Projectives

[LMC & D. Mundici, Projective MV-algebras and rational polyhedra, *Algebra Universalis* Volume 62, Number 1, 63-74]

[LMC & D. Mundici, Rational polyhedra and projective lattice-ordered abelian groups with order unit, *Communications in Contemporary Mathematics* (to appear)]

Introduction

Motivation

Can we characterize the range of $\mathbb{Z}$-maps which domain is some n-cube?

Can we characterize the range of $\mathbb{Z}$-maps which domain is some *n*-cube?

In other words:

Are there intrinsic necessary and sufficient conditions for a rational polyhedron *P* to be equal to $\eta([0,1]^n)$ for some *n* and some $\mathbb{Z}$-map η ?

Combinatorics

Analizing $\mathbb{Z}$-maps

Lemma

Given rational polyhedra $P \subseteq [0,1]^n$ and $Q \subseteq [0,1]^m$ a map $\eta\colon P \to Q$ is a $\mathbb{Z}$-map iff there is a triangulation Δ of P such that over every simplex T of Δ, η coincides with a (affine) linear map η_T with integer coefficients.

Combinatorics

Analizing $\mathbb{Z}$-maps

Combinatorics

Analizing $\mathbb{Z}$-maps

Given a t-simplex $S = \mathrm{conv}(v_0, \ldots, v_t) \subseteq [0,1]^n$ then linear maps from S into $[0,1]^m$ are in one-one correspondence with maps from $\{v_0, \ldots, v_t\}$ into $[0,1]^m$.

Combinatorics

Analizing $\mathbb{Z}$-maps

Let $f\colon [0,1]^n \to \mathbb{R}^m$ be linear map with integer coefficients, i.e. there exists $M \in \mathbb{Z}^{n\times m}$ and $b \in \mathbb{Z}^m$ such that $f(w) = Mw + b$ for each $w \in [0,1]^n$

Combinatorics

Analizing $\mathbb{Z}$-maps

Let $f\colon [0,1]^n \to \mathbb{R}^m$ be linear map with integer coefficients, i.e. there exists $M \in \mathbb{Z}^{n\times m}$ and $b \in \mathbb{Z}^m$ such that $f(w) = Mw + b$ for each $w \in [0,1]^n$

Let $v = (x_1, \dots, x_n) \in [0,1]^n$ be a rational vector.
We define **den**(v) to be the least common denominator of $\{x_1, \dots, x_n\}$, i.e. the smallest $k \in \mathbb{Z}$ such that $kv \in \mathbb{Z}^n$.

Combinatorics

Analizing $\mathbb{Z}$-maps

Let $f\colon [0,1]^n \to \mathbb{R}^m$ be linear map with integer coefficients, i.e. there exists $M \in \mathbb{Z}^{n\times m}$ and $b \in \mathbb{Z}^m$ such that $f(w) = Mw + b$ for each $w \in [0,1]^n$

Let $v = (x_1, \ldots, x_n) \in [0,1]^n$ be a rational vector.
We define **den**(v) to be the least common denominator of $\{x_1, \ldots, x_n\}$, i.e. the smallest $k \in \mathbb{Z}$ such that $kv \in \mathbb{Z}^n$.

Observe that

$$\mathrm{den}(v)f(v) = k(Mv + b) = M(\mathrm{den}(v)v) + \mathrm{den}(v)b \in \mathbb{Z}^m\,.$$

Then $\mathrm{den}(f(v))$ is a divisor of $\mathrm{den}(v)$.

Combinatorics

Regular Simplexes

The vector $\widetilde{v} = \mathrm{den}(v)(v, 1) \in \mathbb{Z}^{n+1}$ is called the **homogeneous correspondent** of v.

Combinatorics

Regular Simplexes

The vector $\widetilde{v} = \mathrm{den}(v)(v, 1) \in \mathbb{Z}^{n+1}$ is called the **homogeneous correspondent** of v.

Definition

A simplex $S \subseteq [0, 1]^n$ is called **regular** if the set of homogeneous correspondents of its vertices is part of a basis of the free abelian group $\mathbb{Z}^{n+1}$.

Combinatorics

Regular Simplexes

Lemma

Let $S = \mathrm{conv}(v_0, \ldots, v_k) \subseteq [0,1]^n$ *be a regular k-simplex, and* $\{w_0, \ldots, w_k\}$ *a set of rational points in* $[0,1]^m$*. Then the following conditions are equivalent:*

(i) *For each* $i = 1, \ldots, k$, $\mathrm{den}(w_i)$ *is a divisor of* $\mathrm{den}(v_i)$*.*

(ii) *For some integer matrix* $M \in \mathbb{Z}^{n\times m}$ *and integer vector* $b \in \mathbb{Z}^m$, $Mv_i + b = w_i$*.*

By a **regular triangulation** of a polyhedron P we understand a triangulation of P consisting of regular simplexes.

Corollary

Let $P \subset [0,1]^n$ be a polyhedron, Δ be a regular triangulation of P and $f : \mathrm{ver}(\Delta) \to ([0,1] \cap \mathbb{Q})^m$ be such that $\mathrm{den}(f(v))$ divides $\mathrm{den}(v)$ for each $v \in \mathrm{ver}(\Delta)$. Then there exists a unique $\mathbb{Z}$-map $\eta\colon P \to [0,1]^m$ satisfying:

1. *η is linear on each simplex of Δ,*
2. *$\eta \restriction \mathrm{ver}(\Delta) = f$.*

Corollary

Let $P \subset [0,1]^n$ be a polyhedron, Δ be a regular triangulation of P and $f : \mathrm{ver}(\Delta) \to ([0,1] \cap \mathbb{Q})^m$ be such that $\mathrm{den}(f(v))$ divides $\mathrm{den}(v)$ for each $v \in \mathrm{ver}(\Delta)$. Then there exists a unique $\mathbb{Z}$-map $\eta\colon P \to [0,1]^m$ satisfying:

1. *η is linear on each simplex of Δ,*
2. *$\eta \upharpoonright \mathrm{ver}(\Delta) = f$.*

Lemma

Let $\eta\colon P \to Q$ be a $\mathbb{Z}$-map there exists a regular triangulation Δ of P such that η is linear over each simplex in Δ.

Combinatorics

Weighted Abstract Simplicial Complexes

For a regular triangulation Δ of a rational polyhedron P, the **skeleton** of Δ is weighted abstract simplicial complex

$$W_\Delta = (\mathcal{V}, \Sigma, \omega)$$

given by the following stipulations:

1. $\mathcal{V}$ = vertices of Δ.
2. For every subset $W = \{w_1, \ldots, w_k\}$ of $\mathcal{V}$, $W \in \Sigma$ iff $\mathrm{conv}(w_1, \ldots, w_k) \in \Delta$.
3. $\omega\colon \mathcal{V} \to \mathbb{N}$, is defined by $\omega(v) = \mathrm{den}(v)$

Combinatorics

Weighted Abstract Simplicial Complexes

Let $\mathfrak{W} = (\mathcal{V}, \Sigma, \omega)$ be a weighted abstract simplicial complex with vertex set $\mathcal{V} = \{v_1, \ldots, v_n\}$.

Let $\mathfrak{W} = (\mathcal{V}, \Sigma, \omega)$ be a weighted abstract simplicial complex with vertex set $\mathcal{V} = \{v_1, \ldots, v_n\}$.

Let $e_1, \ldots, e_n$ the standard basis vectors of $\mathbb{R}^n$, and $\Delta_{\mathfrak{W}}$ be the complex whose vertices are

$$v'_1 = e_1/\omega(v_1), \ldots, v'_n = e_n/\omega(v_n),$$

Combinatorics

Weighted Abstract Simplicial Complexes

Let $\mathfrak{W} = (\mathcal{V}, \Sigma, \omega)$ be a weighted abstract simplicial complex with vertex set $\mathcal{V} = \{v_1, \ldots, v_n\}$.

Let $e_1, \ldots, e_n$ the standard basis vectors of $\mathbb{R}^n$, and $\Delta_{\mathfrak{W}}$ be the complex whose vertices are

$$v'_1 = e_1/\omega(v_1), \ldots, v'_n = e_n/\omega(v_n),$$

and whose k-simplexes ($k = 0, \ldots, n$) are given by

$$\mathrm{conv}(v'_{i(0)}, \ldots, v'_{i(k)}) \in \Delta_W \quad \text{iff} \quad \{v_{i(0)}, \ldots, v_{i(k)}\} \in \Sigma.$$

Combinatorics

Weighted Abstract Simplicial Complexes

Let $\mathfrak{W} = (\mathcal{V}, \Sigma, \omega)$ be a weighted abstract simplicial complex with vertex set $\mathcal{V} = \{v_1, \ldots, v_n\}$.

Let $e_1, \ldots, e_n$ the standard basis vectors of $\mathbb{R}^n$, and $\Delta_{\mathfrak{W}}$ be the complex whose vertices are

$$v'_1 = e_1/\omega(v_1), \ldots, v'_n = e_n/\omega(v_n),$$

and whose k-simplexes ($k = 0, \ldots, n$) are given by

$$\mathrm{conv}(v'_{i(0)}, \ldots, v'_{i(k)}) \in \Delta_W \quad \text{iff} \quad \{v_{i(0)}, \ldots, v_{i(k)}\} \in \Sigma.$$

Then $\Delta_{\mathfrak{W}}$ is a regular triangulation of the polyhedron $|\Delta_{\mathfrak{W}}| \subseteq [0, 1]^n$ called the **canonical realization** of $\mathfrak{W}$.

Given a weighted abstract simplicial complex $\mathfrak{W} = (\mathcal{V}, \Sigma, \omega)$.

Given a weighted abstract simplicial complex $\mathfrak{W} = (\mathcal{V}, \Sigma, \omega)$.

The maps

$$\mathcal{V} \xrightarrow{f} [0,1]^n$$

such that $\mathrm{den}(f(v))$ is a divisor of $\omega(v)$

Given a weighted abstract simplicial complex $\mathfrak{W} = (\mathcal{V}, \Sigma, \omega)$.

The maps

$$\mathcal{V} \xrightarrow{f} [0,1]^n$$

such that $\mathrm{den}(f(v))$ is a divisor of $\omega(v)$

are in one-one correspondence with

Combinatorics

Weighted Abstract Simplicial Complexes

Given a weighted abstract simplicial complex $\mathfrak{W} = (\mathcal{V}, \Sigma, \omega)$.

The maps

$$\mathcal{V} \xrightarrow{f} [0,1]^n$$

such that $\mathrm{den}(f(v))$ is a divisor of $\omega(v)$

are in one-one correspondence with

the $\mathbb{Z}$-maps

$$|\Delta_{\mathfrak{W}}| \xrightarrow{\eta} [0,1]^n$$

that are linear over each simplex of $\Delta_{\mathfrak{W}}$.

Z-images of cubes

Strongly Regular Polyhedra

Definition

A rational polyhedron P is said to be **strongly regular** if there is a regular triangulation Δ of P such that the denominators of the vertices of each maximal simplex of Δ are coprime.

Z-images of cubes

Strongly Regular Polyhedra

Definition

A rational polyhedron P is said to be **strongly regular** if there is a regular triangulation Δ of P such that the denominators of the vertices of each maximal simplex of Δ are coprime.

Lemma

A rational polyhedron P is strongly regular if and only if every regular triangulation Δ of P is such that the denominators of the vertices of each maximal simplex of Δ are coprime.

Z-images of cubes

Strongly Regular Polyhedra

Examples

- For every $n = 1, 2, \ldots$ the n-dimensional cube $[0,1]^n$ is strongly regular.
- Every regular n-simplex $S \subset [0,1]^n$ is strongly regular.

Theorem

Let P and Q be rational polyhedra and $\eta\colon P \to Q$ be a $\mathbb{Z}$-morphism onto Q. If P is a strongly regular then Q is strongly regular.

Z-images of cubes

Theorem

Given a polyhedron $P \subseteq [0,1]^n$ the following conditions are equivalent:

(a) *There exist m and a $\mathbb{Z}$-map $\eta\colon [0,1]^m \to P$ onto P.*

(b) *P satisfies the following conditions:*

1. *P is connected,*
2. *$P \cap \{0,1\}^n \neq \emptyset$, and*
3. *P is strongly regular.*

Z-images of cubes
First Step: Strongly Regular Collapsible Polyhedra

Z-images of cubes

First Step: Strongly Regular Collapsible Polyhedra

Given an abstract simplicial complex $\langle V, \Sigma \rangle$ a simplex $T \in \Sigma$ is said to have a *free face* F if $\emptyset \neq F \subseteq T$ is a facet of T, and if $F \subseteq S \in \Sigma$ then $S = F$ or $S = T$.

Z-images of cubes

First Step: Strongly Regular Collapsible Polyhedra

Given an abstract simplicial complex $\langle V, \Sigma \rangle$ a simplex $T \in \Sigma$ is said to have a *free face* F if $\emptyset \neq F \subseteq T$ is a facet of T, and if $F \subseteq S \in \Sigma$ then $S = F$ or $S = T$.

The transition from $\langle V, \Sigma \rangle$ to the subcomplex $\langle V', \Sigma' = \Sigma \setminus \{T, F\} \rangle$ of $\langle V, \Sigma \rangle$, where $V' = V \setminus F$ if F is a singleton and otherwise $V' = V$ is called an *(abtract) elementary collapse*

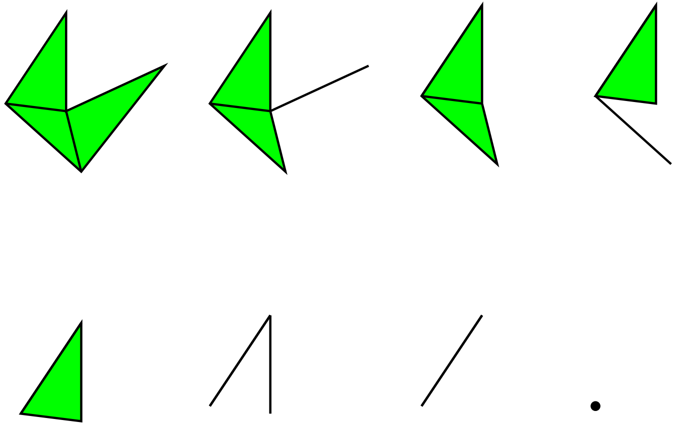
Given an abstract simplicial complex $\langle V, \Sigma \rangle$ a simplex $T \in \Sigma$ is said to have a *free face* F if $\emptyset \neq F \subseteq T$ is a facet of T, and if $F \subseteq S \in \Sigma$ then $S = F$ or $S = T$.

The transition from $\langle V, \Sigma \rangle$ to the subcomplex $\langle V', \Sigma' = \Sigma \setminus \{T, F\} \rangle$ of $\langle V, \Sigma \rangle$, where $V' = V \setminus F$ if F is a singleton and otherwise $V' = V$ is called an *(abtract) elementary collapse*

We say that $\langle V, \Sigma \rangle$ is **collapsible** if it collapses to the abstract simplicial complex consisting of one of its vertices (equivalently any of its vertices).

Z-images of cubes

First Step: Strongly Regular Collapsible Polyhedra



Theorem (Mundici & LMC)

Let $P \subseteq [0,1]^n$ be a polyhedron. Suppose

(i) *P has a collapsible triangulation ∇;*

(ii) *P contains a vertex v of $[0,1]^n$;*

(iii) *P is strongly regular.*

Then there is a $\mathbb{Z}$-map $\eta\colon [0,1]^n \to P$ onto P.

Z-images of cubes

First Step: Strongly Regular Collapsible Polyhedra

Theorem (Mundici & LMC)

Let $P \subseteq [0,1]^n$ be a polyhedron. Suppose

(i) *P has a collapsible triangulation ∇;*

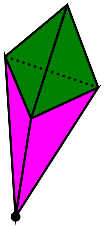
(ii) *P contains a vertex v of $[0,1]^n$;*

(iii) *P is strongly regular.*

Then P is a $\mathbb{Z}$-retract of $[0,1]^n$.

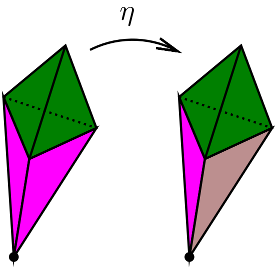
Z-images of cubes

First Step: Strongly Regular Collapsible Polyhedra



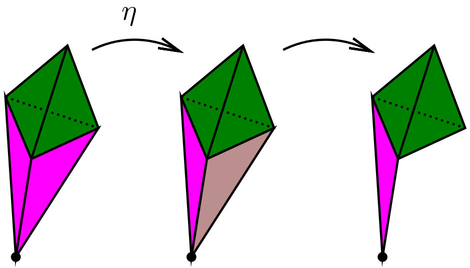
Z-images of cubes

First Step: Strongly Regular Collapsible Polyhedra



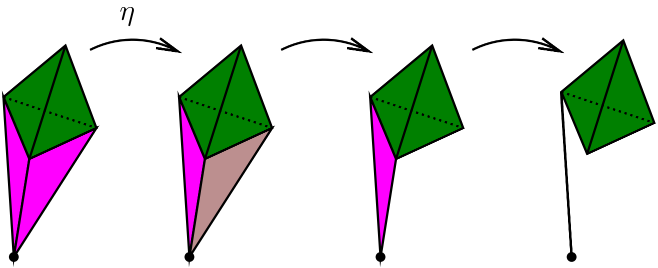
Z-images of cubes

First Step: Strongly Regular Collapsible Polyhedra



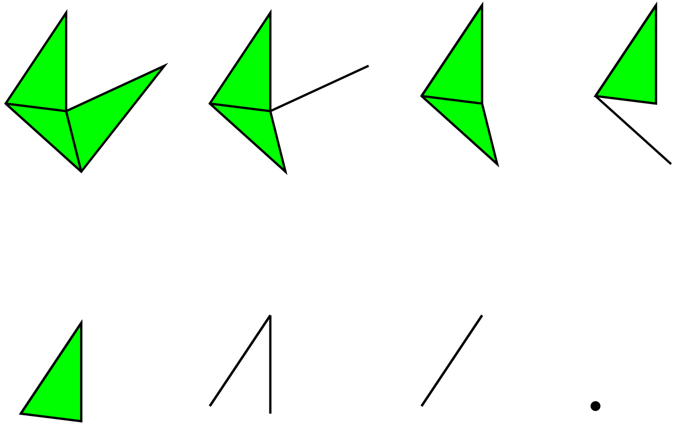
Z-images of cubes

First Step: Strongly Regular Collapsible Polyhedra



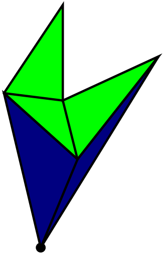
Z-images of cubes

First Step: Strongly Regular Collapsible Polyhedra



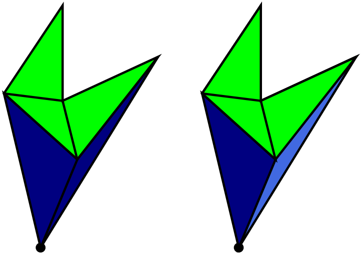
Z-images of cubes

First Step: Strongly Regular Collapsible Polyhedra



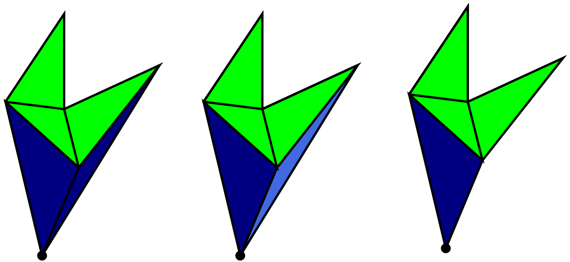
Z-images of cubes

First Step: Strongly Regular Collapsible Polyhedra



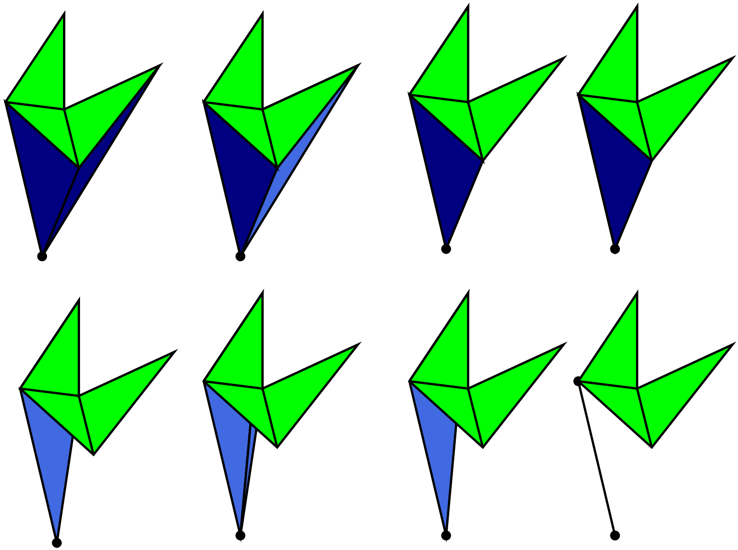
Z-images of cubes

First Step: Strongly Regular Collapsible Polyhedra



Z-images of cubes

First Step: Strongly Regular Collapsible Polyhedra



Z-images of cubes

Second Step: Collapsible Wrapping

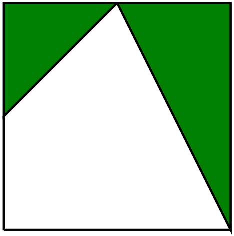
Let $P \subseteq [0,1]^n$ be a rational polyhedron satisfying:

1. P is connected,
2. $P \cap \{0,1\}^n \neq \emptyset$, and
3. P is strongly regular.

Let Δ be a regular triangulation of P.

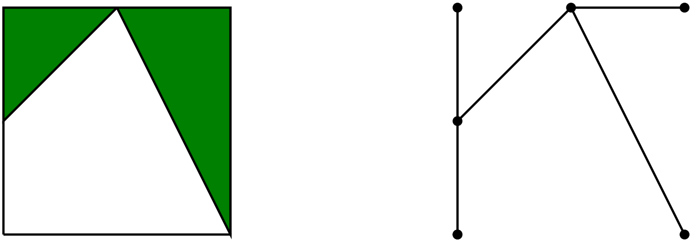
Z-images of cubes

Second Step: Collapsible Wrapping



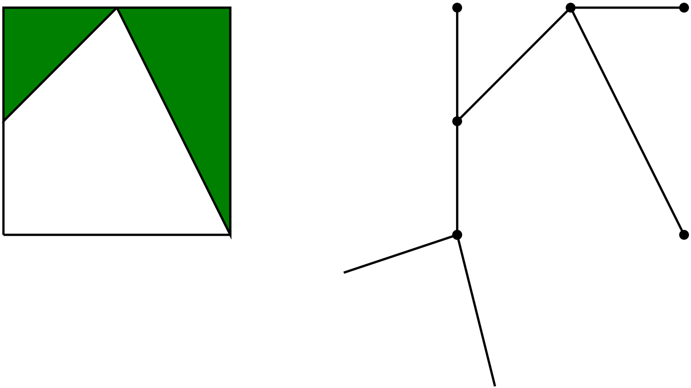
Z-images of cubes

Second Step: Collapsible Wrapping



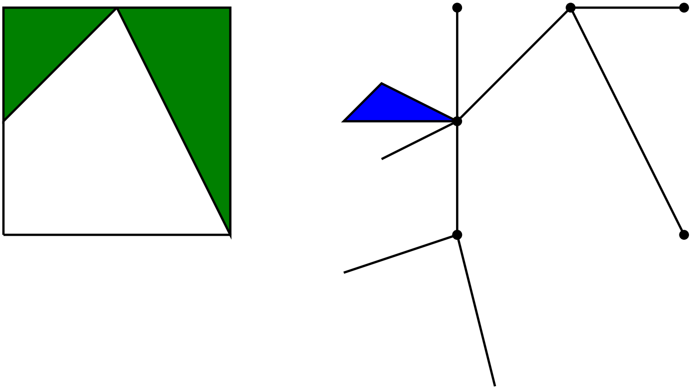
Z-images of cubes

Second Step: Collapsible Wrapping



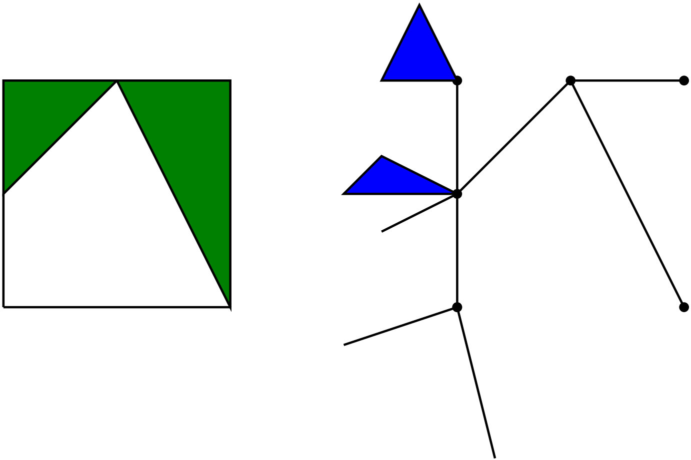
Z-images of cubes

Second Step: Collapsible Wrapping



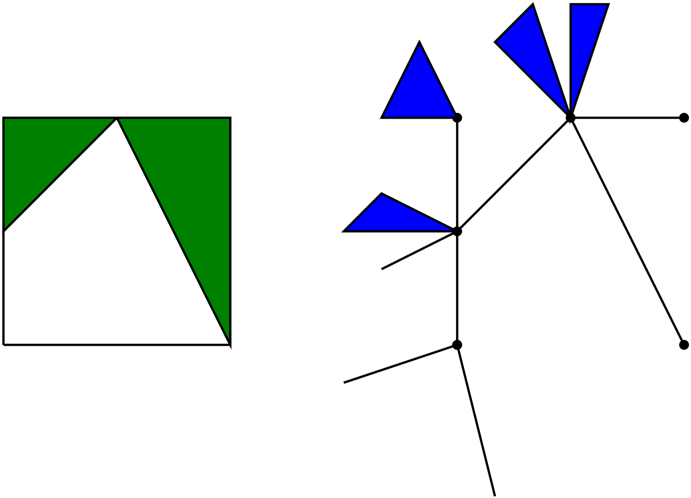
Z-images of cubes

Second Step: Collapsible Wrapping



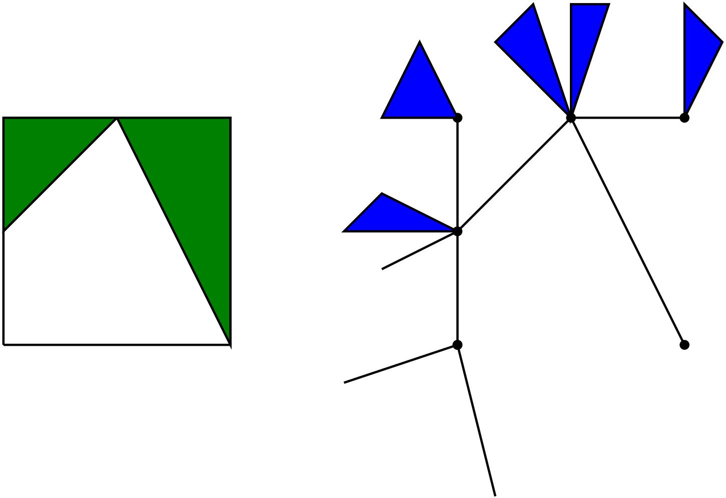
Z-images of cubes

Second Step: Collapsible Wrapping



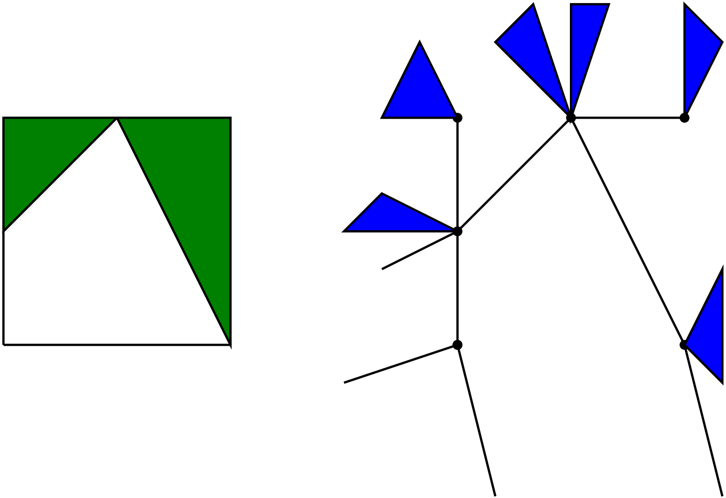
Z-images of cubes

Second Step: Collapsible Wrapping



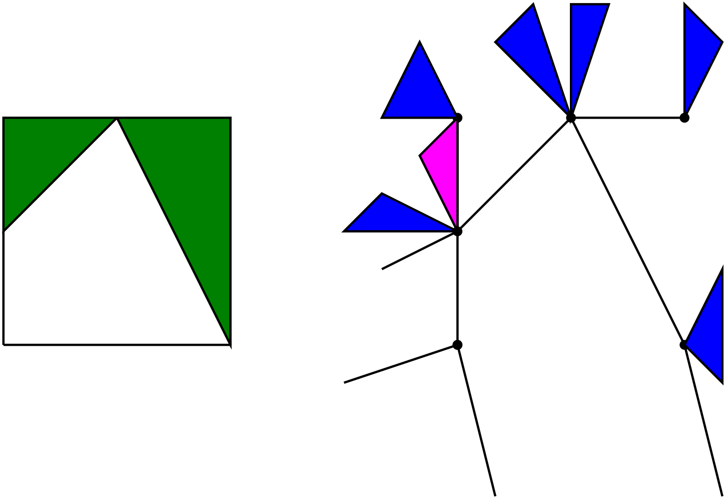
Z-images of cubes

Second Step: Collapsible Wrapping



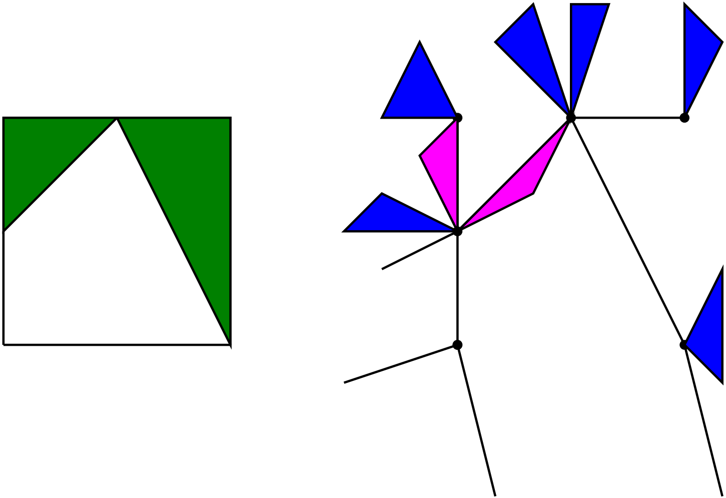
Z-images of cubes

Second Step: Collapsible Wrapping



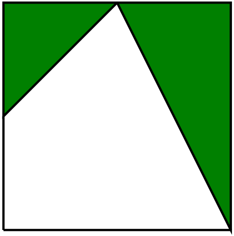
Z-images of cubes

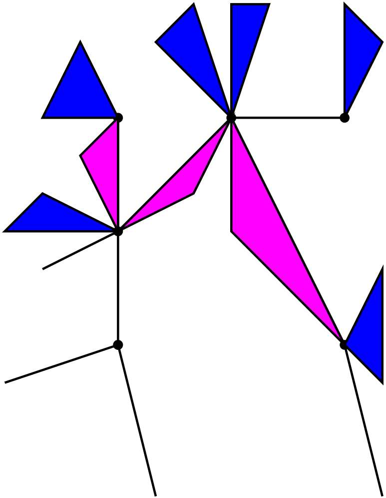
Second Step: Collapsible Wrapping



Z-images of cubes

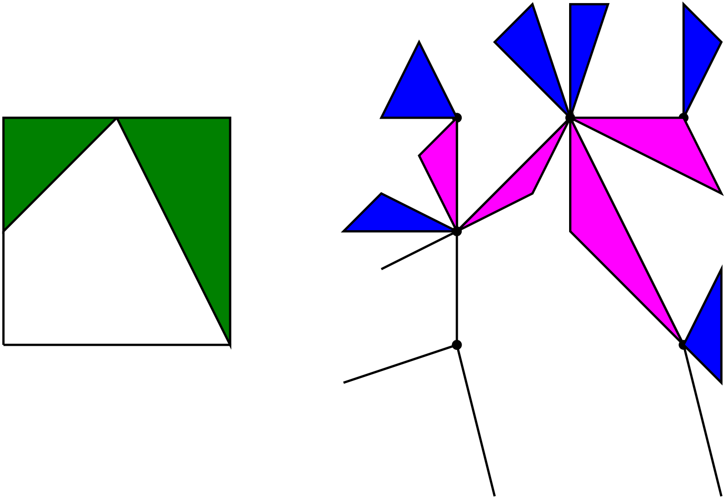
Second Step: Collapsible Wrapping





Z-images of cubes

Second Step: Collapsible Wrapping



Definition

An MV-algebra A is said to be (finitely generated) **weakly projective** if there exist positive integers m and n, and homomorphisms $f\colon \mathcal{M}([0,1]^m) \to A$ and $g\colon A \to \mathcal{M}([0,1]^n)$, such that f is onto A and g is one-one.

Theorem

An MV-algebra is weakly projective if and only if there exist n and a rational polyhedron $P \subseteq [0,1]^n$ satisfying the following conditions:

1. $A \cong \mathcal{M}(P)$,
2. $P \cap \{0,1\}^n \neq \emptyset$,
3. *P is connected, and*
4. *P is a strongly regular.*

Admissible Rules

Admissible Rules

An MV-algebra A is weakly projective if and only if it is finitely presented and $A \in \mathbb{IS}(\mathcal{M}([0,1]^n))$ for some n.

Admissible Rules

An MV-algebra A is weakly projective if and only if it is finitely presented and $A \in \mathbb{IS}(\mathrm{Free}_{MV}(\omega))$.

Admissible Rules

If a pair of formulas φ, ψ with variables $\{x_1, \ldots, x_n\}$ are such that $\mathcal{M}([0,1]^n)\,/\mathrm{cong}(f_\varphi, f_\psi) \in \mathbb{ISP}_U(\mathrm{Free}_{MV}(\omega))$,

Admissible Rules

If a pair of formulas φ, ψ with variables $\{x_1, \dots, x_n\}$ are such that $\mathcal{M}([0,1]^n) / \mathrm{cong}(f_\varphi, f_\psi) \in \mathbb{ISP}_U(\mathrm{Free}_{MV}(\omega))$, then:

$$\varphi \approx \psi \models_{\mathrm{Free}_{MV}(\omega)} \{\alpha_i \approx \beta_i \mid i = 1, \dots, m\}$$

if and only if there exist $j \in \{1, \dots, m\}$ such that

$$\varphi \approx \psi \models_{MV} \alpha_j \approx \beta_j$$

Admissible Rules

If a pair of formulas φ, ψ with variables $\{x_1, \ldots, x_n\}$ are such that $\mathcal{M}([0,1]^n)\,/\mathrm{cong}(f_\varphi, f_\psi) \in \mathbb{ISP}_U(\mathrm{Free}_{MV}(\omega))$, then:

$$\varphi \approx \psi \models_{\mathrm{Free}_{MV}(\omega)} \{\alpha_i \approx \beta_i \mid i = 1, \ldots, m\}$$

if and only if there exist $j \in \{1, \ldots, m\}$ such that

$$\varphi \approx \psi \models_{MV} \alpha_j \approx \beta_j$$

if and only if

$$\varphi \leftrightarrow \psi \vdash_{Ł_\infty} \alpha_j \leftrightarrow \beta_j$$

Admissible Rules

If a formula φ with variables $\{x_1, \ldots, x_n\}$ is such that $\mathcal{M}([0,1]^n)/\mathrm{cong}(f_\varphi, 1) \in \mathbb{ISP}_U(\mathrm{Free}_{MV}(\omega))$, then:

$$\varphi \approx \varphi \to \varphi \models_{\mathrm{Free}_{MV}(\omega)} \{\alpha_i \approx \alpha_i \to \alpha_i \mid i = 1, \ldots, m\}$$

if and only if there exist $j \in \{1, \ldots, m\}$ such that

$$\varphi \approx \varphi \to \varphi \models_{MV} \alpha_j \approx \alpha_j \to \alpha_j$$

if and only if

$$\varphi \vdash_{\text{Ł}_\infty} \alpha_j$$

Admissible Rules

Admissible Saturated Formulas

[E. Jeřábek, Bases of Admissible Rules of Łukasiewicz Logic, *Journal of Logic Computation* 20 (6), 1149-1163 (2010)]

Admissible Rules

Admissible Saturated Formulas

[E. Jeřábek, Bases of Admissible Rules of Łukasiewicz Logic, *Journal of Logic Computation* 20 (6), 1149-1163 (2010)]

Definition

A formula φ is admissibly saturated in a logic L if for every finite set Δ of formulas, $\varphi \mid\!\sim_L \Delta$ implies $\varphi \mid\!\sim_L \psi$ for some $\psi \in \Delta$.

Admissible Rules

Admissible Saturated Formulas

If φ is a formula such that $\mathcal{M}([0,1]^n)\,/\mathrm{cong}(f_\varphi, 1)$ is weakly projective, then φ is admissible saturated (in $Ł_\infty$).

If φ is a formula such that $\mathcal{M}([0,1]^n)/\mathrm{cong}(f_\varphi, 1)$ is weakly projective, then φ is admissible saturated (in Ł$_\infty$).

Theorem (E. Jeřábek)

A formula φ with variables in $\{x_1, \ldots, x_n\}$ is admissibly saturated in Ł$_\infty$ if and only if

(i) $f_\varphi^{-1}(1) \cap \{0,1\}^n \neq \emptyset$,

(ii) $f_\varphi^{-1}(1)$ *is connected, and*

(iii) $f_\varphi^{-1}(1)$ *is a finite union of anchored polytopes.*

Theorem

And MV-algebra is weakly projective if and only if there exist n and a rational polyhedron $P \subseteq [0,1]^n$ satisfying the following conditions:

1. $A \cong \mathcal{M}(P)$,
2. $P \cap \{0,1\}^n \neq \emptyset$,
3. *P is connected, and*
4. *P is a strongly regular.*

A formula φ with variables in $\{x_1, \ldots, x_n\}$ is admissibly saturated in Ł$_\infty$

if and only if

$\mathcal{M}([0,1]^n)\ /\mathrm{cong}(f_\varphi, 1)$ is weakly projective.

Summarizing

- Images of $\mathbb{Z}$-maps.

Summarizing

- Images of $\mathbb{Z}$-maps.
- Weakly projective MV-Algebras.

Summarizing

- Images of $\mathbb{Z}$-maps.
- Weakly projective MV-Algebras.
- Relation with admissible saturated formulas.

Thank you!

leonardo.cabrer@math.unibe.ch