A modal logic for belief functions on MV-algebras

Tommaso Flaminio Lluís Godo Enrico Marchioni

Artificial Intelligence Research Institute
(IIIA - CSIC), Campus UAB,
Spain.

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Outline

1. Belief functions
2. Logical approach: $FP(\Lambda_k,Ł)$ and $FP(C\Lambda_k,Ł)$
3. Semantics for the modal logics $FP(\Lambda_k,Ł)$ and $FP(C\Lambda_k,Ł)$
   - Probabilistic models
   - Belief function models

Belief functions on Boolean algebras

Let X be a finite set (the *frame of discernment*) and let $m : \mathcal{P}(X) \to [0, 1]$ be a map such that

$$\sum_{A \subseteq X} m(A) = 1, \text{ and } m(\emptyset) = 0.$$

The map m is called the *mass assignment*, and the *belief function* over $\mathcal{P}(X)$ defined from m is the map $\mathbf{b}_m : \mathcal{P}(X) \to [0, 1]$ such that for every $A \in \mathcal{P}(X)$

$$\mathbf{b}_m(A) = \sum_{B \subseteq A} m(B).$$

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A subset $A \subseteq X$ such that $m(A) > 0$ is said to be a *focal element*, and clearly the belief function $\mathbf{b}_m$ is defined from the restriction of m over the focal elements.

Notice that every mass assignment m on $\mathcal{P}(X)$ induces a probability measure $\mathbf{P}_m$ on $\mathcal{P}(\mathcal{P}(X))$. Therefore, given a mass assignment m, for every $A \subseteq X$, we can equivalently define

$$\mathbf{b}_m(A) = \mathbf{P}_m(\beta_A),$$

where $\beta_A = \{B \mid B \subseteq A\}$, or as membership function on $\mathcal{P}(\mathcal{P}(X))$

$$\beta_A : B \in \mathcal{P}(X) \mapsto \begin{cases} 1 & \text{if } B \subseteq A \\ 0 & \text{otherwise,} \end{cases}$$

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Belief functions on MV-algebras (1)

In order to generalize belief functions to MV-algebras of functions, Kroupa provides the following approach: Consider a finite set X and let M be the MV-algebra of functions $[0,1]^X$ (i.e. fuzzy subsets of X). For every $a \in M$, let $\hat{\rho}_a : \mathcal{P}(X) \to [0,1]$ be defined as follows: for every $B \subseteq X$,

$$\hat{\rho}_a(B) = \min\{a(x) : x \in B\}.$$

The map $\hat{\rho}_a$ generalizes β_A because if a is a Boolean function, then $\hat{\rho}_a = \beta_a$.

Definition

A Kroupa belief function is a map $\hat{\mathbf{b}} : [0,1]^X \to [0,1]$ such that, for every $a \in [0,1]^X$,

$$\hat{\mathbf{b}}(a) = \hat{\mathbf{s}}(\hat{\rho}_a),$$

where $\hat{\mathbf{s}} : [0,1]^{\mathcal{P}(X)} \to [0,1]$ is a state.

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- Since X is finite, one can equivalently define

$$\hat{\mathbf{b}}(a) = \sum_{B \subseteq X} \hat{\rho}_a(B) \cdot \hat{\mathbf{s}}(B).$$

- The restriction of $\hat{\mathbf{s}}$ to $\mathcal{P}(X)$ (call it $\hat{m}$) is a *classical mass assignment*. Therefore a *focal element* is any $B \subseteq X$ such that $\hat{m}(B) > 0$. That is, focal elements are *classical sets*.

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Belief functions on MV-algebras (2)

We generalize Kroupa's belief functions on $[0,1]^X$ by allowing focal elements to be elements of the same MV-algebra $[0,1]^X$. What we need to generalize is the map ρ that measures the degree of inclusion between fuzzy sets.

For every $a \in [0,1]^X$ we define $\rho_a : [0,1]^X \to [0,1]$ as follows: for every $b \in [0,1]^X$,

$$\rho_a(b) = \min\{b(x) \Rightarrow a(x) : x \in X\}.$$

For every $a \in [0,1]^X$, the map ρ_a generalizes $\hat{\rho}_a$ because for every crisp subset B of X, $\rho_a(B) = \hat{\rho}_a(B)$.

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Kroupa approach	Our approach
$\hat{\rho}_a$ measures the degree of inclusion of a crisp set B in the fuzzy set a	ρ_a measures the degree of inclusion of a fuzzy set b in the fuzzy set a
Crisp evidence: B is a crisp set	Fuzzy evidence: b is a fuzzy set.
$\hat{\mathbf{b}}(a) = \hat{\mathbf{s}}(\hat{\rho}_a)$ and $\hat{\mathbf{s}} : [0,1]^{\mathcal{P}(X)} \to [0,1]$	$\mathbf{b}(a) = \mathbf{s}(\rho_a)$ and $\mathbf{s} : [0,1]^{[0,1]^X} \to [0,1]$

Logical approach: $FP(\Lambda_k,Ł)$ and $FP(C\Lambda_k,Ł)$

The above definitions suggest that a logic for belief functions on MV-events can be introduced by expanding the language of Łukasiewicz logic by two unary modalities:

- A modality $\Box$ whose interpretation is intended to capture the behavior of the measure of inclusion $\hat{\rho}$, or ρ, we are dealing with.
- A modality Pr that respects the axioms of states on MV-algebras.

Finally we interpret the *belief* of φ by $\Pr(\Box\varphi)$ (a similar approach was used by Godo, Hájek and Esteva to deal with belief functions over Boolean events).

Consider the k-valued Łukasiewicz logic expanded with rational truth constants $Ł_k^{\mathbf{c}}$.

A $Ł_k^{\mathbf{c}}$-*Kripke model* is a triple $\langle W, e, R\rangle$ where:

- W is a non-empty set of possible worlds,
- for every possible world w, $e(\cdot, w)$ is a truth-evaluation of $Ł_k^{\mathbf{c}}$ into S_k,
- $R : W \times W \to S_k$ is an accessibility relation.

We denote by Fr the class of $Ł_k^{\mathbf{c}}$-Kripke models.

If the accessibility relation R is crisp (i.e. $R : W \times W \to \{0, 1\}$), then the model is called a *classical Kripke model*, and we will denote by CFr the class of all classical Kripke models.

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The modal logics Λ_k and $C\Lambda_k$

Bou, Esteva, Godo and Rodriguez introduce the logics $\Lambda(Fr, Ł_k^{\mathbf{c}})$ and $\Lambda(CFr, Ł_k^{\mathbf{c}})$ by enlarging the language of $Ł_k^{\mathbf{c}}$ by a unary modality $\Box$, and defining well formed formulas as usual. Now we are going to consider the two fragments Λ_k and $C\Lambda_k$ of $\Lambda(Fr, Ł_k^{\mathbf{c}})$ and $\Lambda(CFr, Ł_k^{\mathbf{c}})$, whose well formed formulas have unnested occurrences of $\Box$, so to keep the modal logic to be locally finite.

Given a formula ϕ, and a (classical, $Ł_k^{\mathbf{c}}$)-Kripke model $K = \langle W, e, R \rangle$, for every $w \in W$, we define the truth value of ϕ in K at w as follows:

- If ϕ is a formula of $Ł_k^{\mathbf{c}}$, then $\|\phi\|_w = e(\phi, w)$,
- If $\phi = \Box\psi$, then $\|\Box\psi\|_w = \bigwedge_{w' \in W} (R(w, w') \Rightarrow \|\psi\|_{w'})$,
- If ϕ is a compound formula, its truth value is computed the truth functionality.

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Then the logic Λ_k has the following axioms:

1. all the axioms for $Ł_k^{\mathbf{c}}$;
2. $\Box 1$;
3. $(\Box\varphi \wedge \Box\psi) \to \Box(\varphi \wedge \psi)$;
4. $\Box(\bar{r} \to \varphi) \leftrightarrow (\bar{r} \to \Box\varphi)$.

The rules of Λ_k are Modus Ponens, $\{\varphi, \varphi \to \psi\} \vdash \psi$; and Monotonicity, $\varphi \to \psi \vdash \Box\varphi \to \Box\psi$.

The logic $C\Lambda_k$ is Λ_k plus the axiom $\{\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)\}$.

- The logic Λ_k is sound and complete w.r.t. *Fr*.
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Probabilistic logics over Λ_k, and $C\Lambda_k$

The logics $FP(\Lambda_k,Ł)$ and $FP(C\Lambda_k,Ł)$ have a language obtained by expanding the language of Λ_k by a unary modality σ. Formulas are those of Λ_k, plus the class $\mathfrak{F}^\sigma$ that includes $\mathfrak{F}^\Box$ and satisfies the following: for every $\psi \in \mathfrak{F}^\Box$, $\sigma\psi \in \mathfrak{F}^\sigma$, and $\mathfrak{F}^\sigma$ is closed under the connectives of Ł.

Axioms and rules of $FP(\Lambda_k,Ł)$ are as follows:

1. All the axioms and rules of Λ_k restricted to the formulas in $\mathfrak{F}^\Box$;
2. The following axioms for σ (cf. [FG07]):
   1. $\sigma\top$.
   2. $\sigma(\neg\varphi) \leftrightarrow \neg\sigma(\varphi)$.
   3. $\sigma(\varphi \oplus \psi) \leftrightarrow [(\sigma(\varphi) \to \sigma(\psi\&\varphi)) \to \sigma(\psi)]$.
3. The rule of Necessitation, $\varphi \vdash \sigma(\varphi)$.

Axioms and rules of $FP(C\Lambda_k,Ł)$ are as above, replacing the axioms of Λ_k for the formulas in $\mathfrak{F}^\Box$, with those of $C\Lambda_k$.

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Probabilistic models

The first kind of models for $FP(\Lambda_k,Ł)$ and $FP(C\Lambda_k,Ł)$ are defined as follows:

Definition

A *probabilistic Kripke model* is a system

$$M = \langle W, e, R, s\rangle$$

such that its reduct $\langle W, e, R\rangle$ is a $Ł_k^{\mathbf{c}}$-Kripke model, and $s : \mathfrak{F}_M^{\Box} \to [0, 1]$ is a state, where $\mathfrak{F}_M^{\Box} = \{\|\varphi\|_M : w \in W \mapsto \|\varphi\|_{M,w} : \varphi \in \mathfrak{F}^{\Box}\}$.

A probabilistic $Ł_k^{\mathbf{c}}$-Kripke model such that its reduct $\langle W, e, R\rangle$ is a classical Kripke model, is called a *probabilistic classical Kripke frame*.

Let $M = \langle W, e, R, s \rangle$ be a probabilistic $Ł_k^{\mathbf{c}}$ (classical) Kripke model. For every $\Phi \in \mathfrak{F}^\sigma$, and for every $w \in W$, we define the truth value of Φ in M at w inductively as follows:

- If $\Phi \in \mathfrak{F}^\Box$, then its truth value $\|\Phi\|_{M,w}$ is evaluated in the fragment $\langle W, e, R \rangle$ as we defined in the previous section.
- If $\Phi = \sigma\psi$, then $\|\sigma\psi\|_{M,w} = s(\|\psi\|_M)$.
- If Φ is a compound formula, its truth values is computed by truth functionality.

Theorem (Probabilistic completeness)

(1) The logic $FP(\Lambda_k, Ł)$ is sound and finitely strong complete with respect to the class of probabilistic $Ł_k^{\mathbf{c}}$-Kripke models.
(2) The logic $FP(C\Lambda_k, Ł)$ is sound and finitely strong complete with respect to the class of probabilistic classical Kripke models.

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Belief function models

Definition

The set of *belief formulas* (or *B-formulas*) is the subset of $\mathfrak{F}^\sigma$ defined as follows: *atomic belief formulas* are those in the form $\sigma\Box\psi$ (where of course ψ is a formula in $Ł_k^{\mathbf{c}}$), that will be henceforth denoted by $B(\psi)$; *compound belief formulas* are defined from atomic ones using the connectives of Ł. The set of belief formulas will be denoted by $\mathfrak{F}^B$.

Let now Ω be the set of all the evaluations of $Ł_k^{\mathbf{c}}$, over the (finite) set of propositional variables V, i.e. $\Omega = (S_k)^V$. For every formula φ without occurrences of modalities (i.e. φ is a formula in the language of $Ł_k^{\mathbf{c}}$), let $\|\varphi\|_\Omega : \Omega \to S_k$ be defined as $\|\varphi\|_\Omega(w) = w(\varphi)$.

Definition

A *(Kroupa) belief function model* is a pair $N = (\Omega, m)$ where Ω is as above, and $m : (S_k)^\Omega \to [0,1]$ ($m : (\{0,1\})^\Omega \to [0,1]$) satisfies $\sum_{f \in (S_k)^\Omega} m(f) = 1$, and $m(\emptyset) = 0$. Then the corresponding belief function bel_m is defined as usual: for every formula φ,

$$bel_m(\varphi) = \sum_{g \in (S_k)^\Omega} \rho_{\|\varphi\|_\Omega}(g) \cdot m(g).$$

For every belief formula Φ, and every belief function model $N = (\Omega, m)$, Φ is evaluated into N a follows:

- If $\Phi = B(\varphi)$ is atomic, then $\|B(\varphi)\|_N = bel_m(\varphi)$.
- If Φ is compound, then $\|\Phi\|_N$ is computed by truth functionality as usual.

Theorem

Let Φ be a belief formula. Then for every (Kroupa) belief function model $D = (\Omega, m)$ there exists a (classical) probabilistic Kripke model $K = (W, e, R, s)$ such that Φ, $\|\Phi\|_M = \|\Phi\|_D$, and vice-versa.

Hence, if we limit to belief formulas, and belief theories, then $FP(\Lambda_k, Ł)$ is sound and finitely complete with respect to the class of belief models. An analogous result holds for $FP(C\Lambda_k, Ł)$ with respect to Kroupa belief models.

Theorem

Let Φ be a belief formula. Then for every (Kroupa) belief function model $D = (\Omega, m)$ there exists a (classical) probabilistic Kripke model $K = (W, e, R, s)$ such that Φ, $\|\Phi\|_M = \|\Phi\|_D$, and vice-versa.

Hence, if we limit to belief formulas, and belief theories, then $FP(\Lambda_k, Ł)$ is sound and finitely complete with respect to the class of belief models. An analogous result holds for $FP(C\Lambda_k, Ł)$ with respect to Kroupa belief models.

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