On involutive FL_e-Algebras

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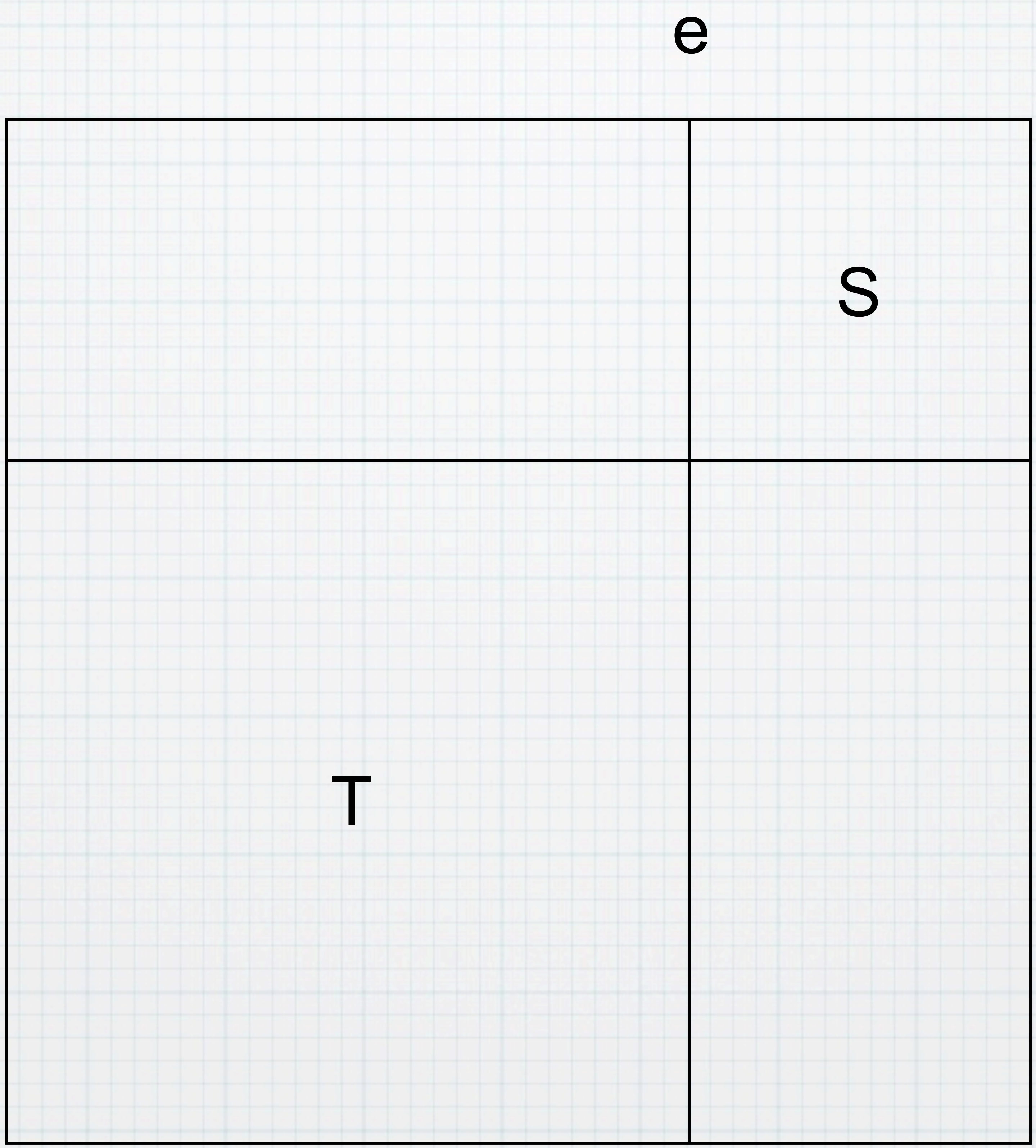
Terminology

* Commutative partially ordered monoids will be referred to as uninorms.
* Uninorms which are integral (resp. dually integral) will be referred to as t-norms (resp. t-conorms).

e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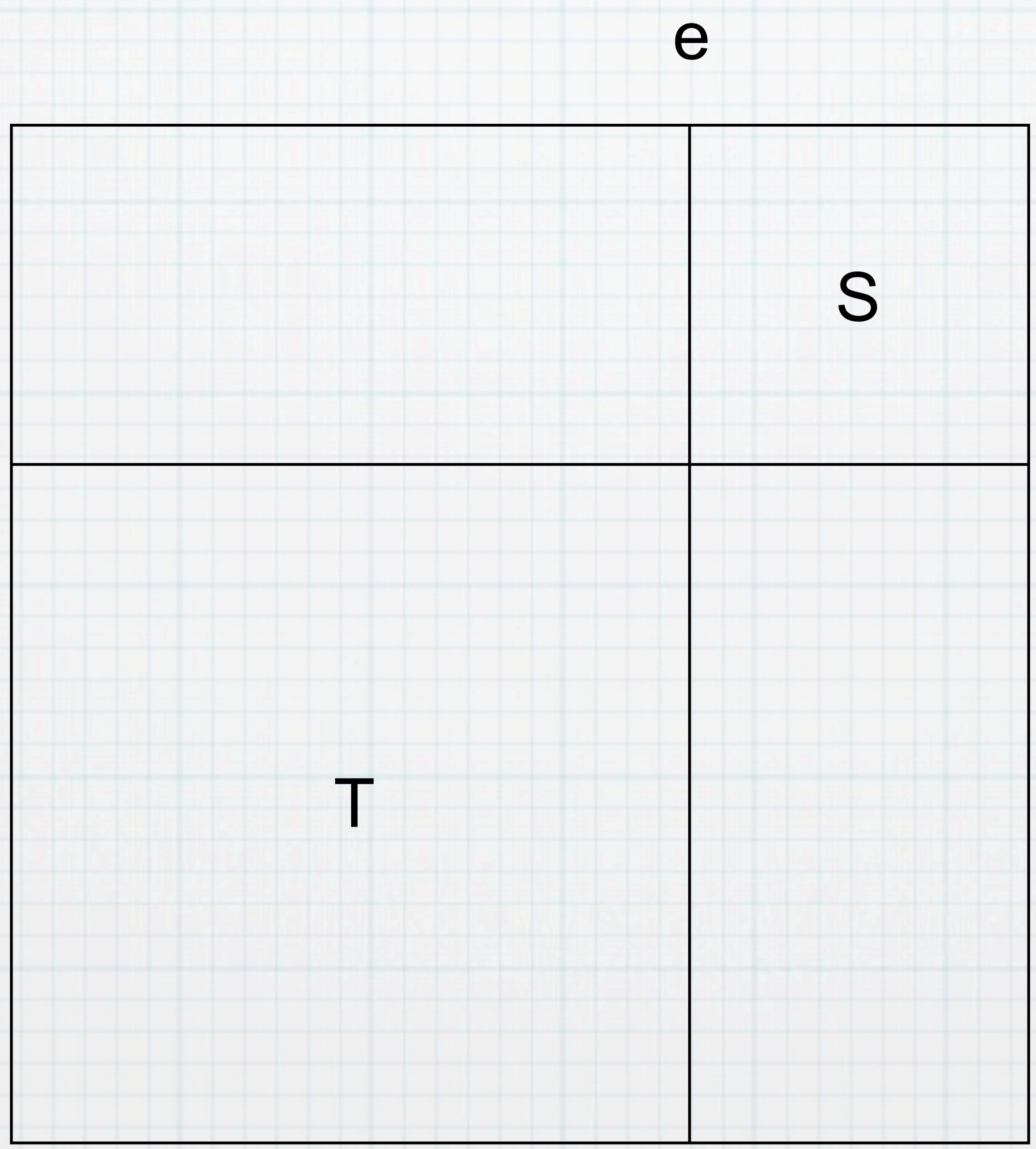
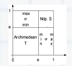
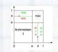
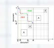
What is known about uninorms?



If T and S are continuous







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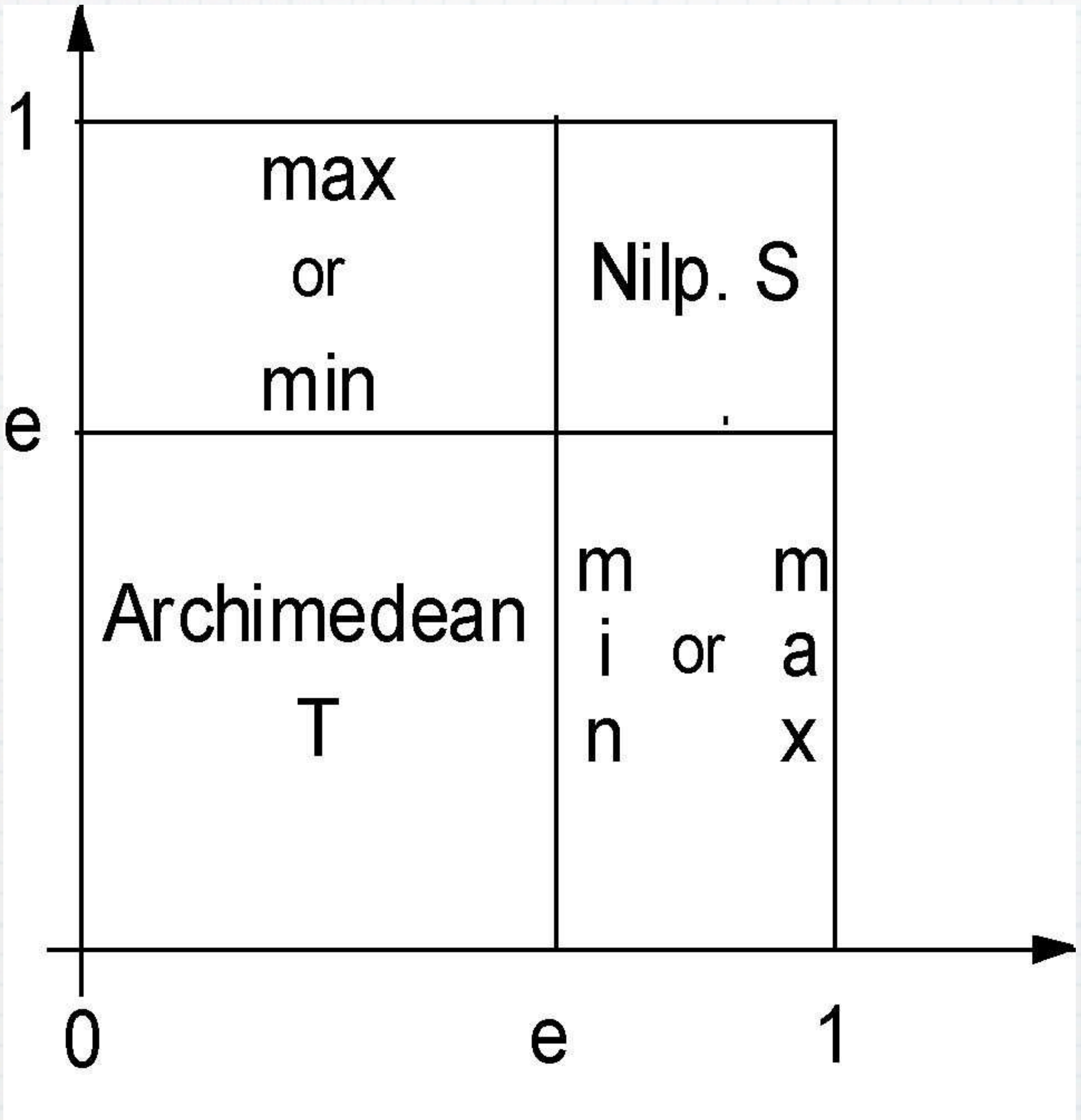
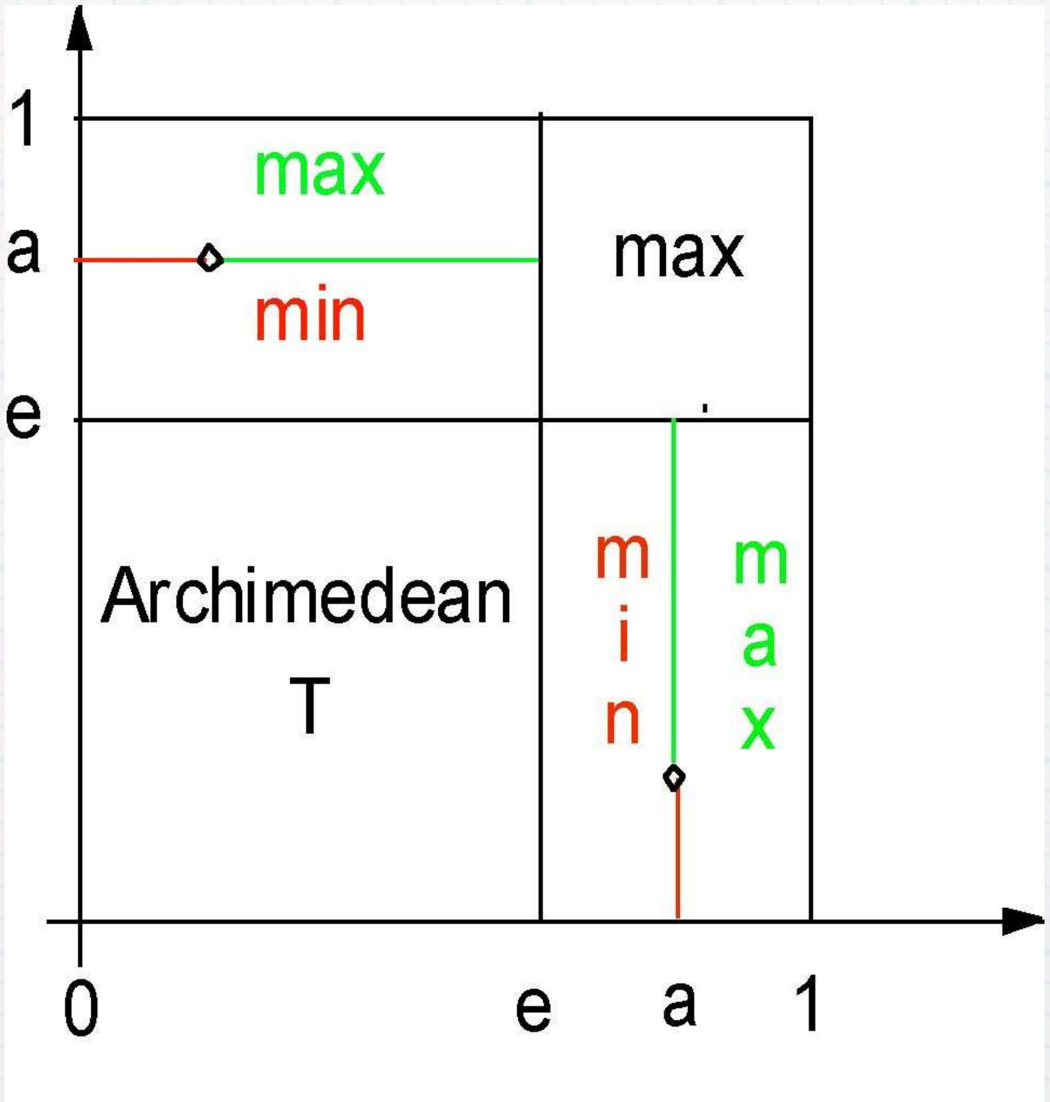


Courtesy: Fodor

If T and S are continuous



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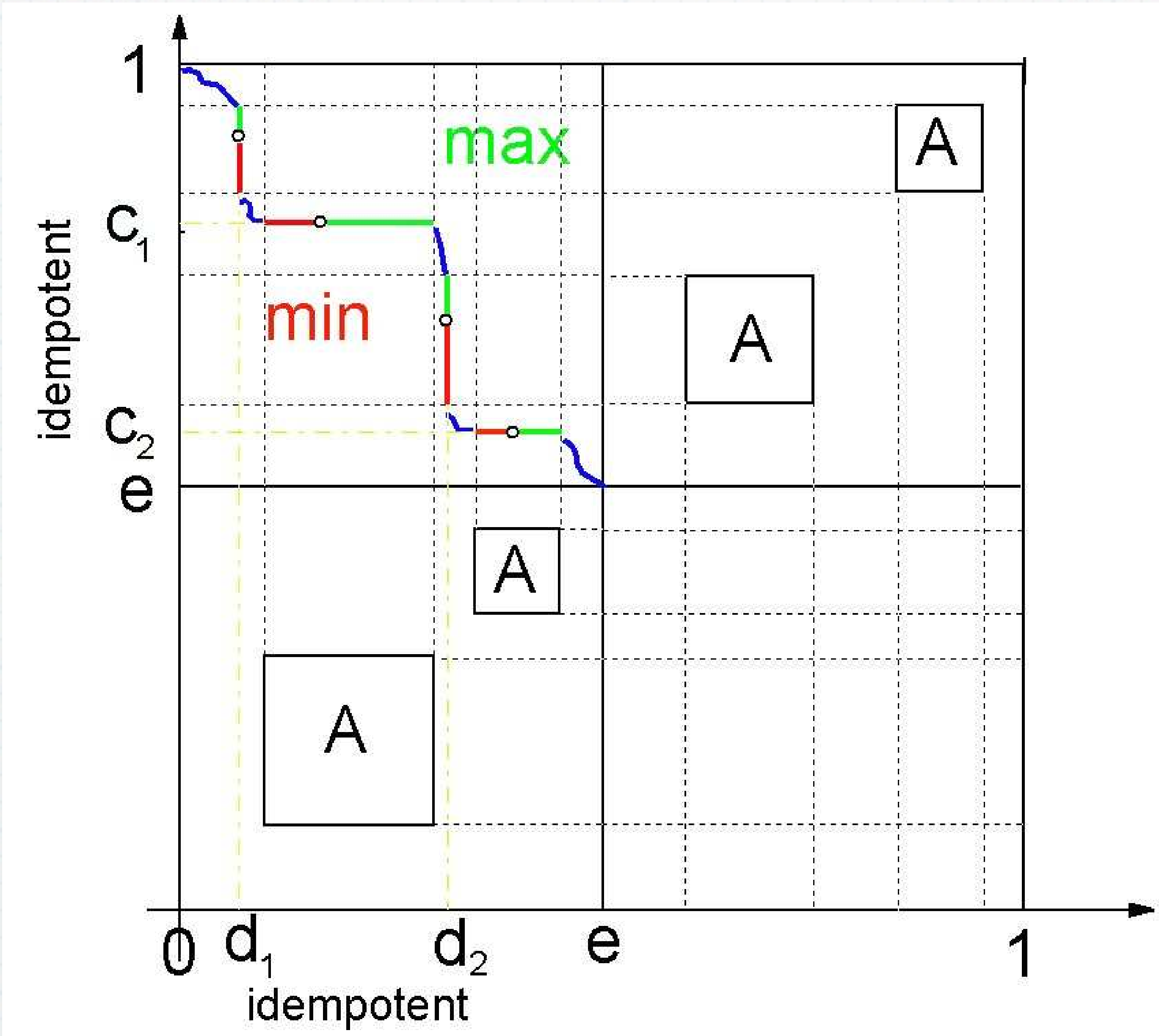
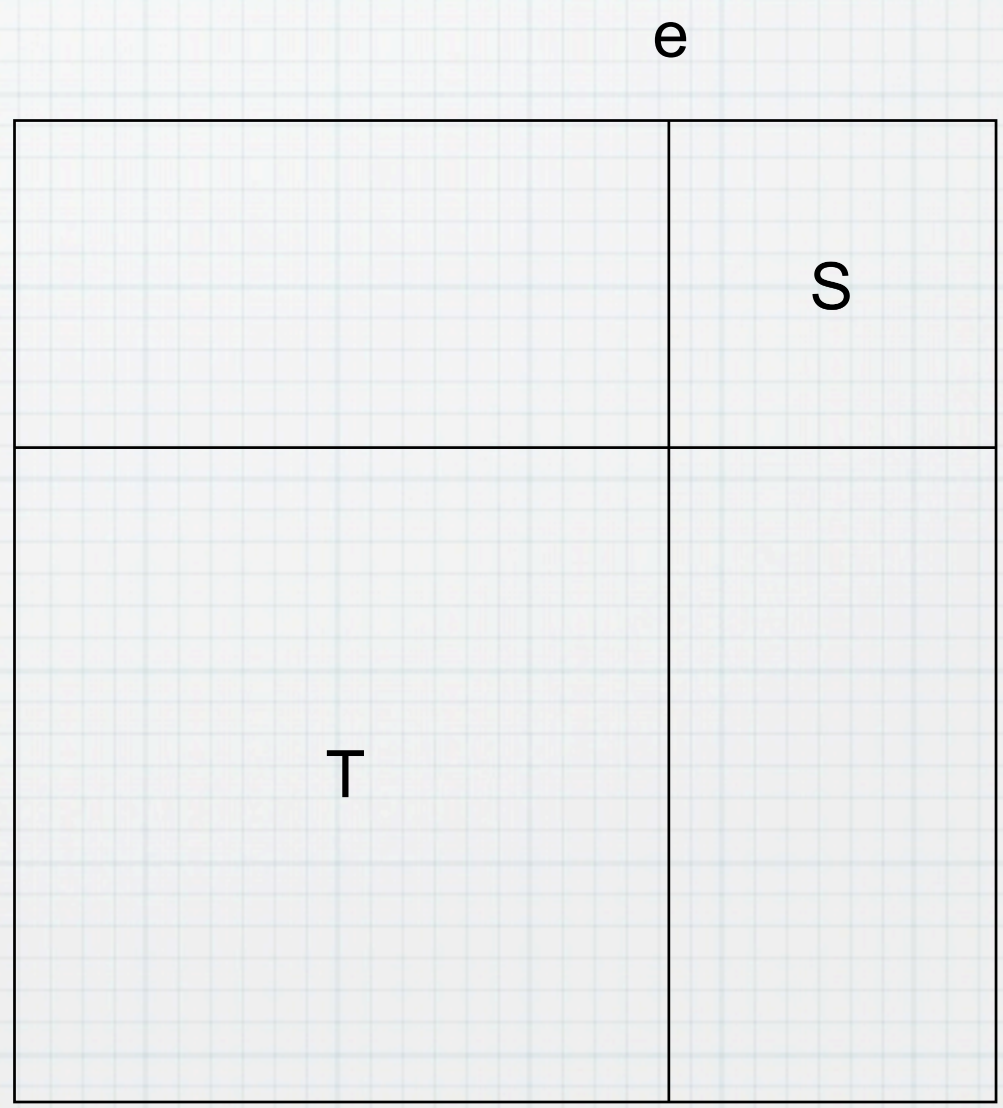


Courtesy: Fodor

What if T and S are only residuated (left-continuous)?

Q1: Structural description
Q2: Classification



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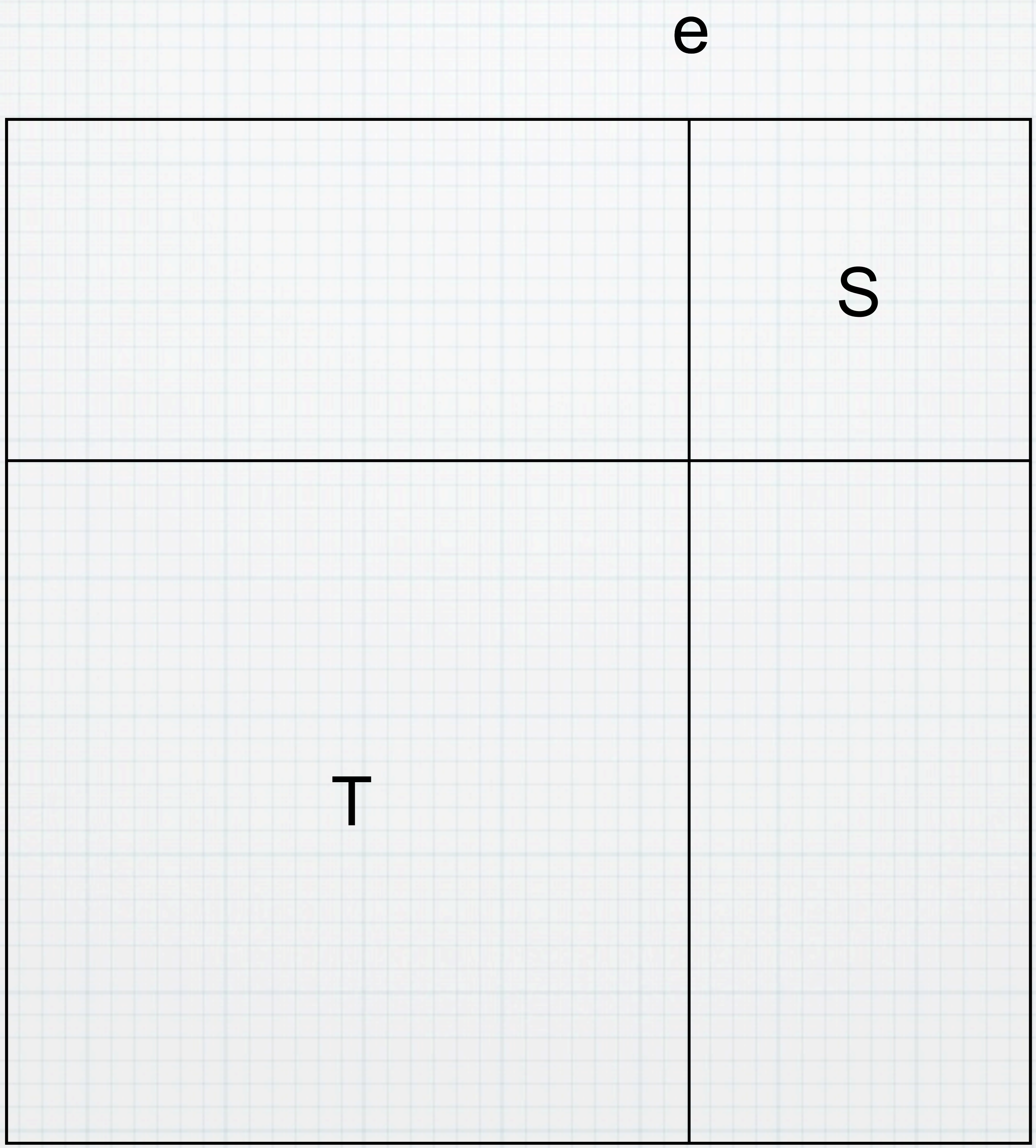
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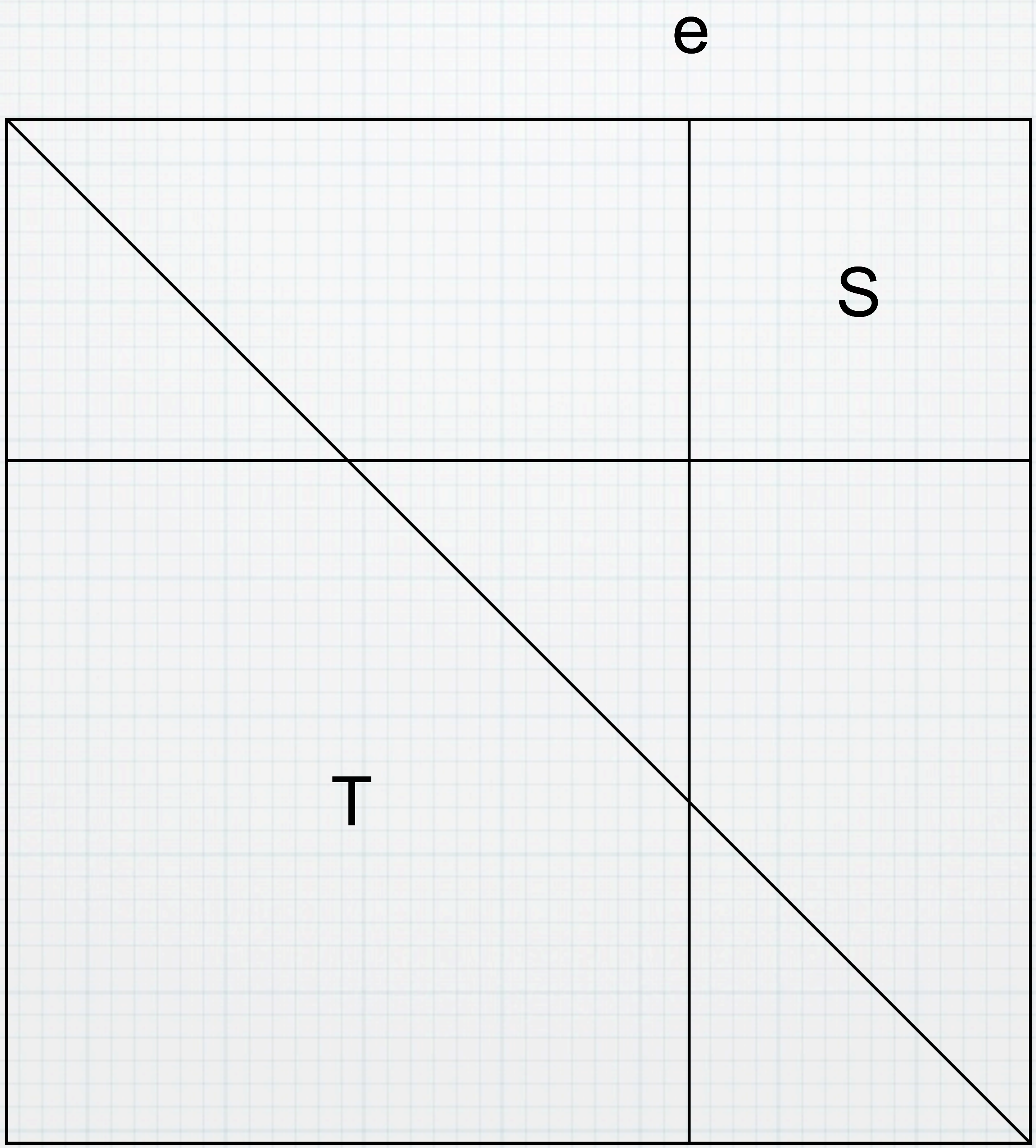
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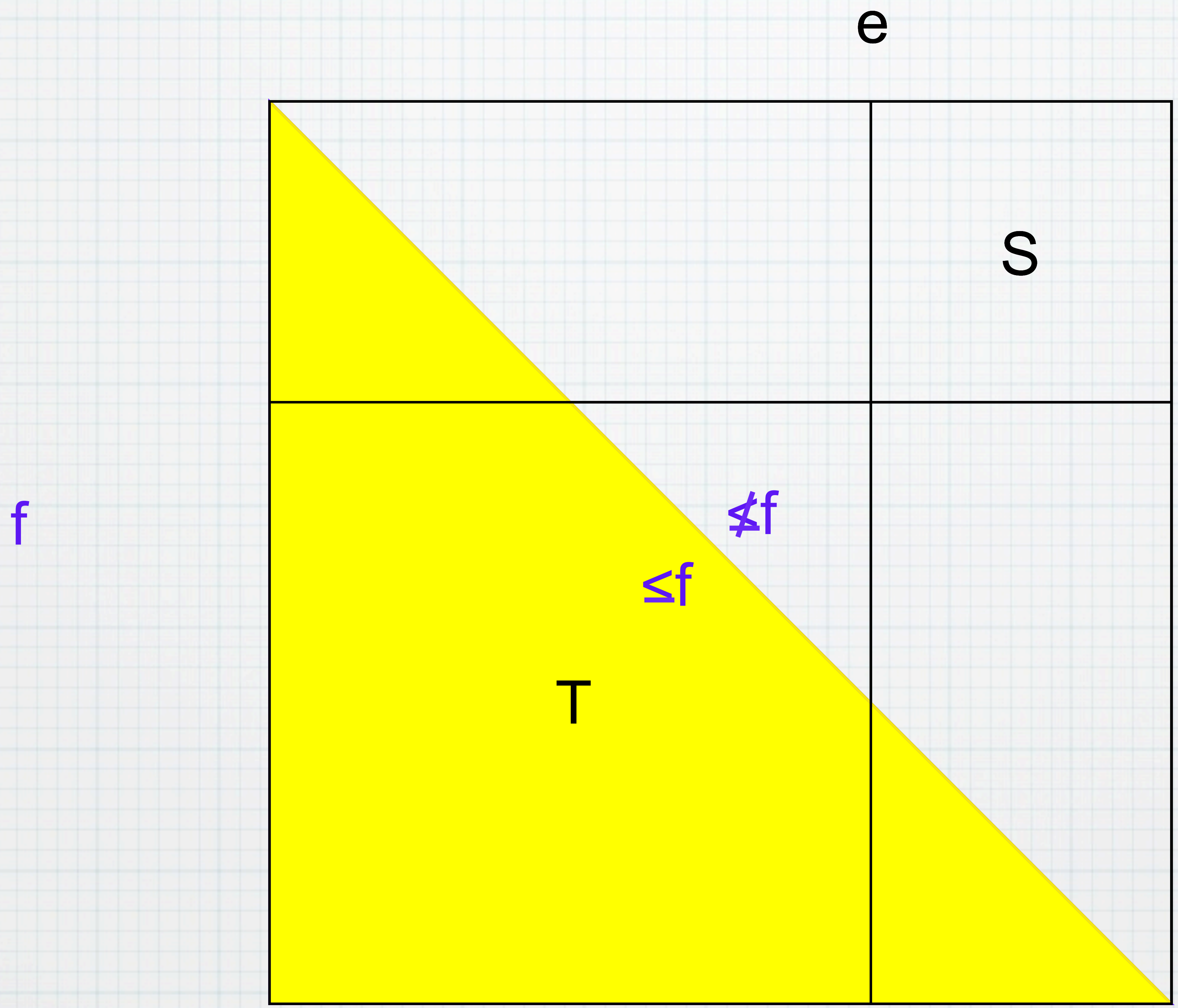
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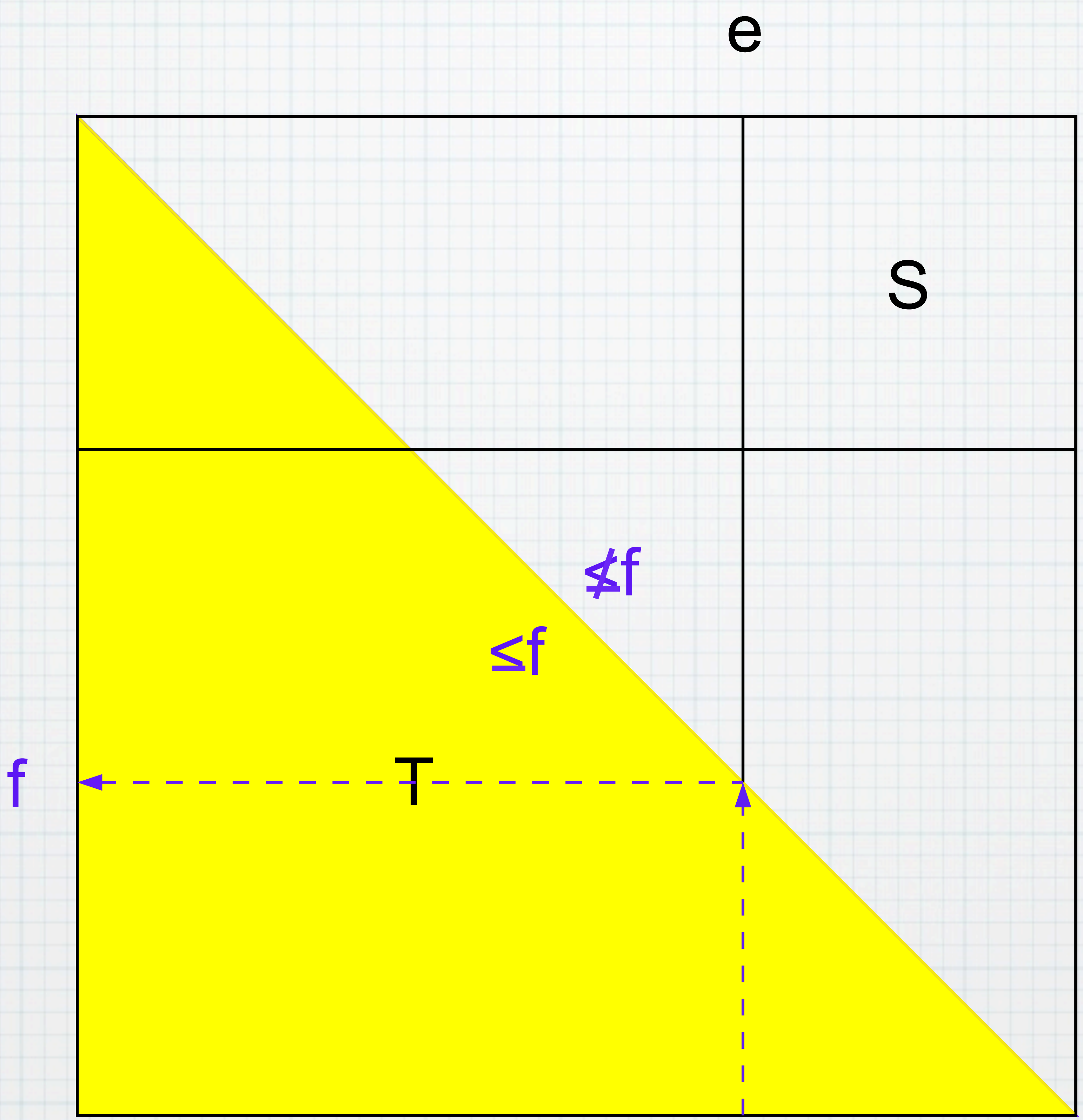
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Basic Definition

Definition 1 $\mathcal{U} = \langle X, \circledast, \leq, e, f \rangle$ is called an *involutive FL_e-algebra* if

1. $\mathcal{C} = \langle X, \leq \rangle$ is a poset,
2. $\circledast$ is a uninorm over $\mathcal{C}$ with neutral element e,
3. for every $x \in X$, $x \rightarrow_{\circledast} f = \max\{z \in X \mid x \circledast z \leq f\}$ exists, and
4. for every $x \in X$, we have $(x \rightarrow_{\circledast} f) \rightarrow_{\circledast} f = x$.

We will call $\circledast$ an involutive uninorm. It is not difficult to see that every involutive uninorm is residuated and isotone. Therefore, $' : X \rightarrow X$ given by

$$x' = x \rightarrow_{\circledast} f$$

is an order-reversing involution.

Basic Definition

ASubL 4.pdf (page 2 of 19)

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Denote

$$X^+ = \{x \in X \mid x \geq e\} \quad \text{and} \quad X^- = \{x \in X \mid x \leq e\}.$$

Overview

* Motivation beyond the algebraic interest
* Twin rotation construction
* Complete, densely ordered chains
* Finite Chains

Motivation beyond the algebraic interest

substructural fuzzy logics Metcalfe Montagna.pdf (page 3 of 30)

DEFINITION 1. **MAILL** *consists of the following axioms and rules:*

(L1) $A \to A$

(L2) $(A \to B) \to ((B \to C) \to (A \to C))$

(L3) $(A \to (B \to C)) \to (B \to (A \to C))$

(L4) $((A \odot B) \to C) \leftrightarrow (A \to (B \to C))$

(L5) $(A \wedge B) \to A$

(L6) $(A \wedge B) \to B$

(L7) $((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$

finite involutive uninorms.pdf

[17] G. Metcalfe, F. Montagna, *Substructural fuzzy logics*, Journal of Symbolic Logic, 72(3): 834–864, 2007.

(L12) $\bot \to A$

(L13) $A \to \top$

$$\frac{A \quad A \to B}{B}\ (mp) \qquad\qquad \frac{A \quad B}{A \wedge B}\ (adj)$$

DEFINITION 2. Uninorm logic **UL** *is* **MAILL** *plus:*

$$(PRL) \quad (A \to B) \wedge t) \vee ((B \to A) \wedge t)$$

Motivation beyond the algebraic interest

substructural fuzzy logics Metcalfe Montagna.pdf (page 3 of 30)

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(L6) & (A \wedge B) \to B \\
(L7) & ((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C)) \\
(L8) & A \to (A \vee B) \\
(L9) & B \to (A \vee B) \\
(L10) & ((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C) \\
(L11) & A \leftrightarrow (t \to A) \\
(L12) & \bot \to A \\
(L13) & A \to \top
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substructural fuzzy logics Metcalfe Montagna.pdf (page 3 of 30)

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DEFINITION 2. Uninorm logic **UL** *is* **MAILL** *plus:*

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- Involutive uninorm logic **IUL** *is* **UL** *plus* $(INV)\ \neg\neg A \to A$.

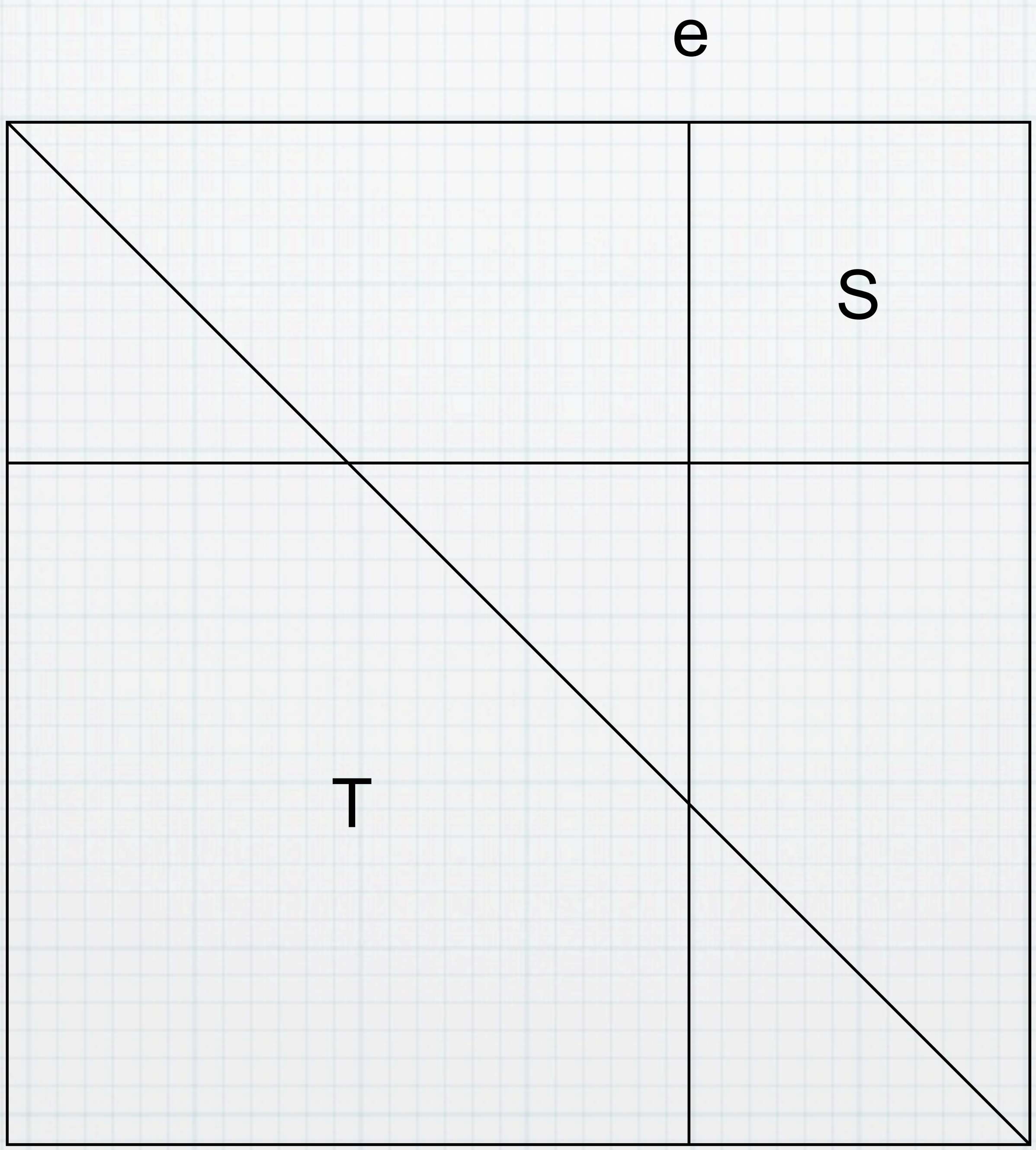
Motivation beyond the algebraic interest

$$\text{Logics} = \{\mathbf{UL}, \mathbf{IUL}, \mathbf{MTL}, \mathbf{IMTL}, \mathbf{G}, \mathbf{UML}, \mathbf{IUML}\}.$$

substructural fuzzy logics Metcalfe Montagna.pdf (page 4 of 30)

- Involutive uninorm logic **IUL** *is* **UL** *plus* (INV) $\neg\neg A \to A$.
- Monoidal t-norm logic **MTL** *is* **UL** *plus* (W) $(A \to t) \wedge (f \to A)$.
- Involutive monoidal t-norm logic **IMTL** *is* **MTL** *plus* (INV).
- Gödel logic **G** *is* **MTL** *plus* (ID) $A \leftrightarrow (A \odot A)$.
- Uninorm mingle logic **UML** *is* **UL** *plus* (ID).
- Involutive uninorm mingle logic **IUML** *is* **IUL** *plus* (ID) *and* (FP) $t \leftrightarrow f$.

Twin rotation



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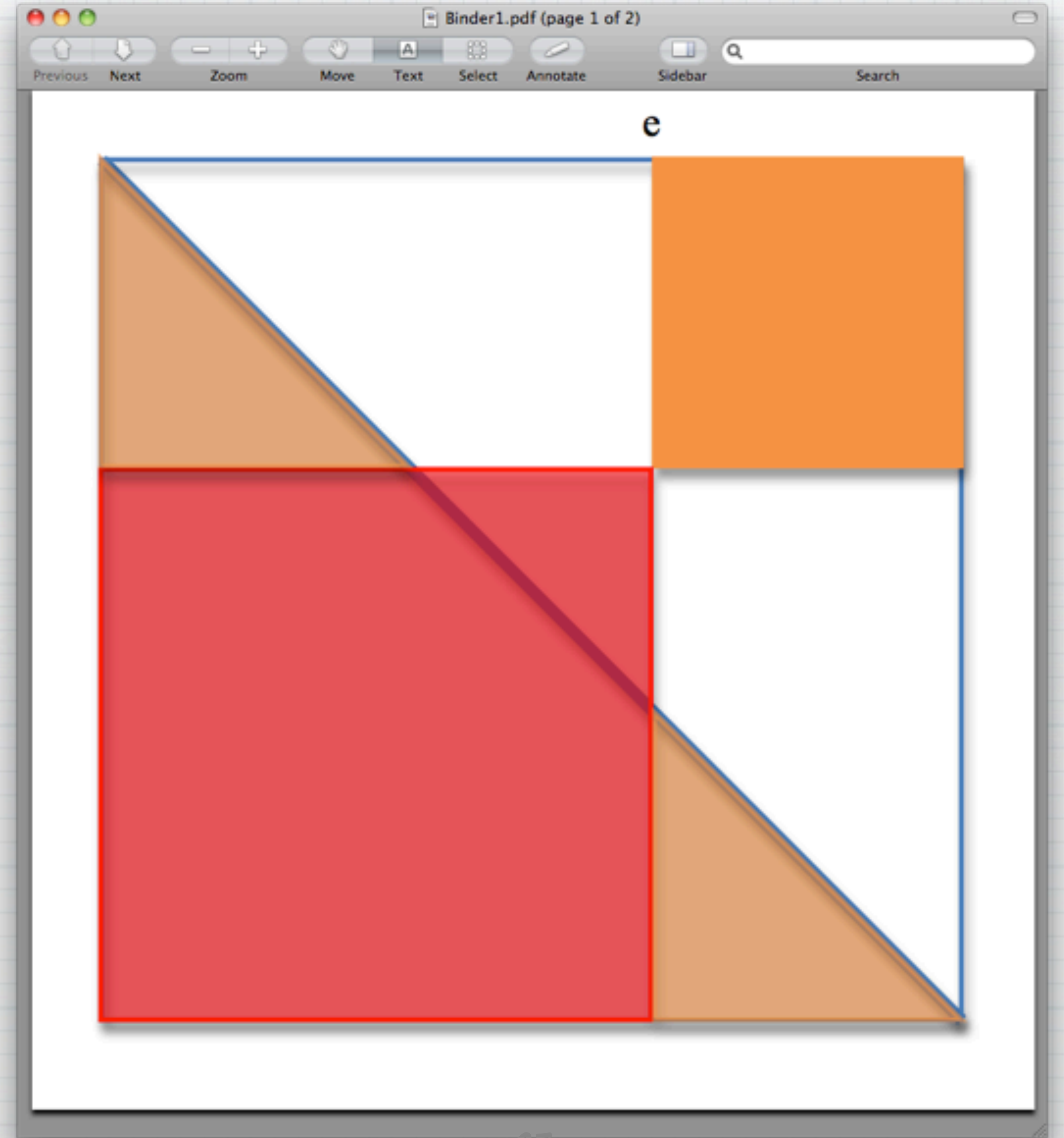

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Definition 4 (Twin-rotation construction) Let X_1 be a partially ordered set with top element t, and and X_2 be a partially ordered set with bottom element t such that the connected ordinal sum $os_c\langle X_1, X_2\rangle$ of X_1 and X_2 (that is putting X_1 under X_2, and identifying the top of X_1 with the bottom of X_2) has an order reversing involution $'$. Let $\otimes$ and $\oplus$ be commutative, residuated semigroups on X_1 and X_2, respectively, both with neutral element t. Assume, in addition, that

1. in case $t' \in X_1$ we have $x \to_\otimes t' = x'$ for all $x \in X_1$, $x \geq t'$, and
2. in case $t' \in X_2$ we have $x \to_\oplus t' = x'$ for all $x \in X_2$, $x \leq t'$.

Denote

$$\mathcal{U}_\otimes^\oplus = \langle os_c\langle X_1, X_2\rangle, \circledast, \leq, t, f\rangle$$

where $f = t'$ and $\circledast$ is defined as follows:

$$x \circledast y = \begin{cases} x \otimes y & \text{if } x, y \in X_1 \\ x \oplus y & \text{if } x, y \in X_2 \\ (x \to_\oplus y')' & \text{if } x \in X_2,\ y \in X_1, \text{ and } x \leq y' \\ (y \to_\oplus x')' & \text{if } x \in X_1,\ y \in X_2, \text{ and } x \leq y' \\ (y \to_\otimes (x' \wedge t))' & \text{if } x \in X_2,\ y \in X_1, \text{ and } x \not\leq y' \\ (x \to_\otimes (y' \wedge t))' & \text{if } x \in X_1,\ y \in X_2, \text{ and } x \not\leq y' \end{cases} . \quad (12)$$

Call $\circledast$ (resp. $\mathcal{U}_\otimes^\oplus$) the twin-rotation of $\otimes$ and $\oplus$ (resp. of the first and the second partially ordered monoid).

Twin rotation

Twin Rotation.pdf (1 page)

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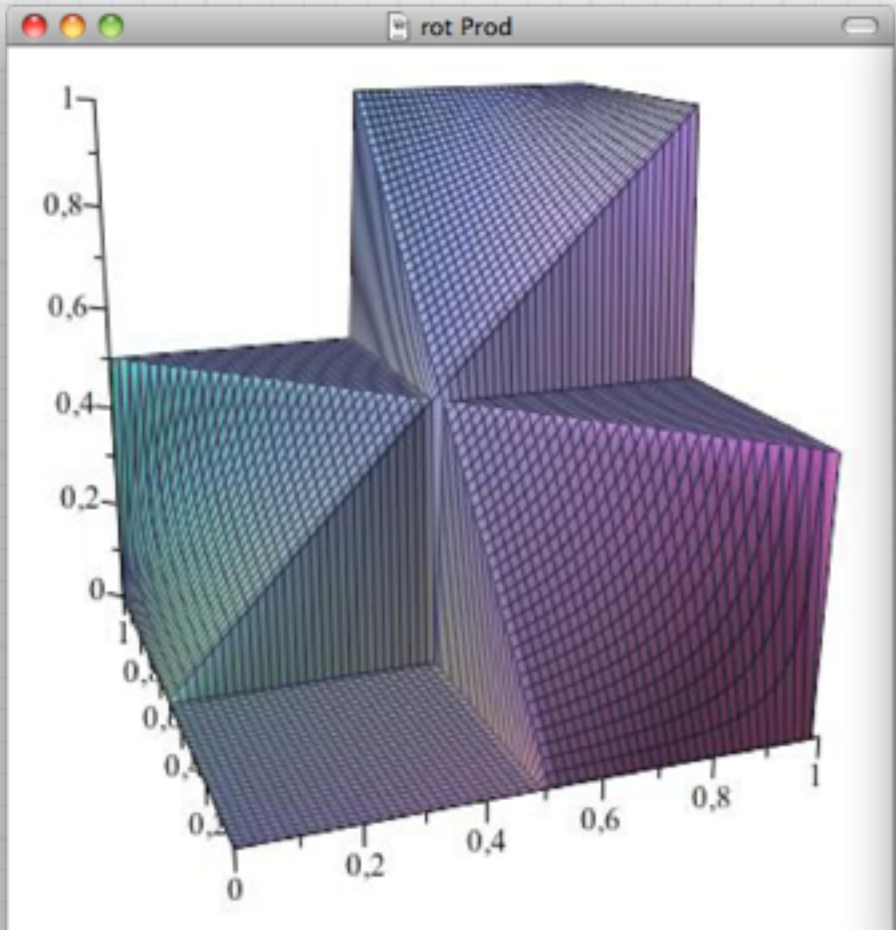

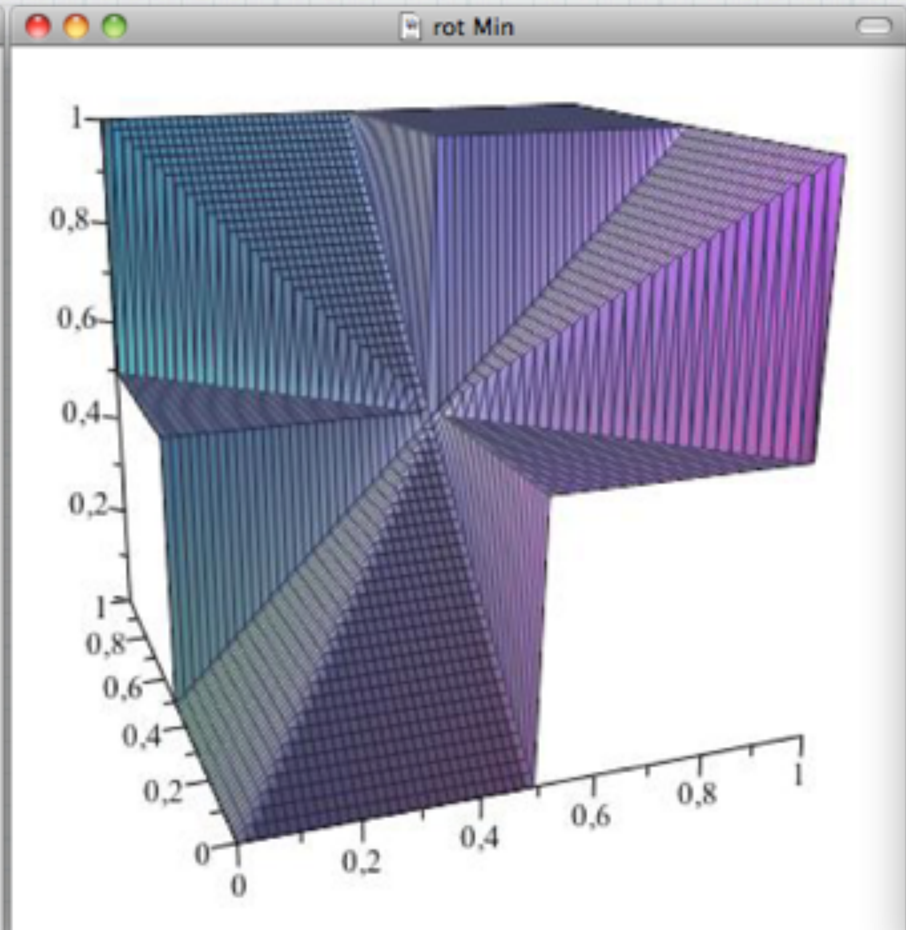

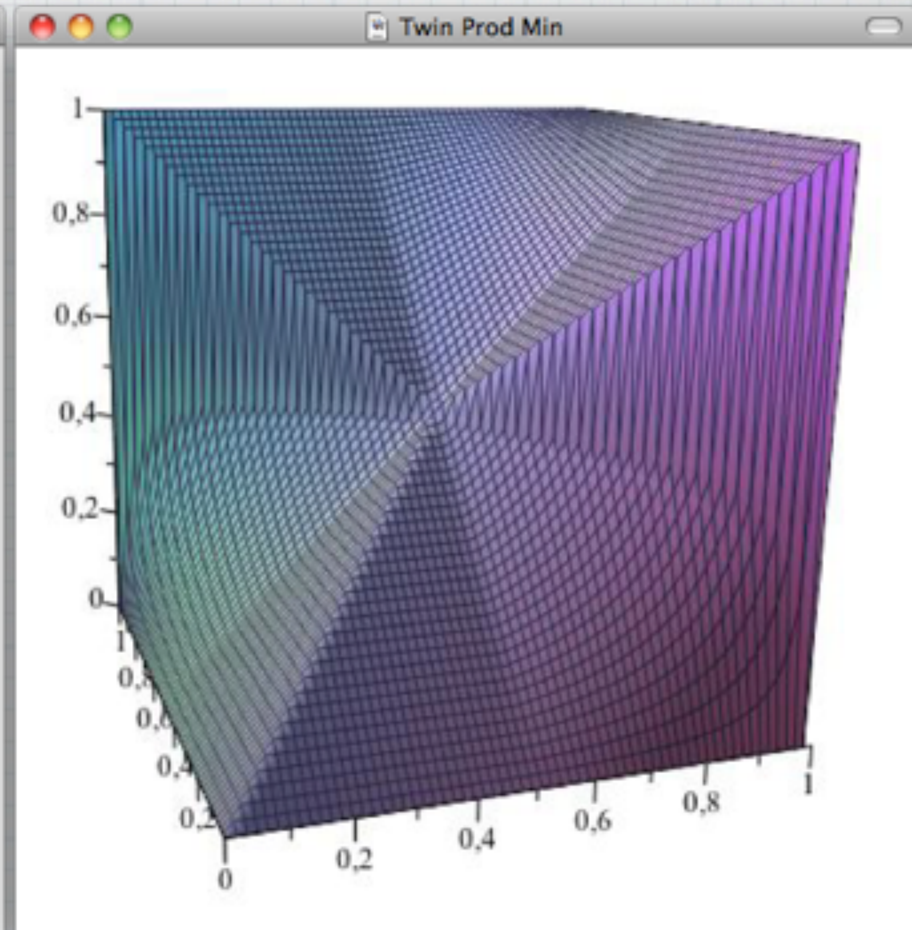

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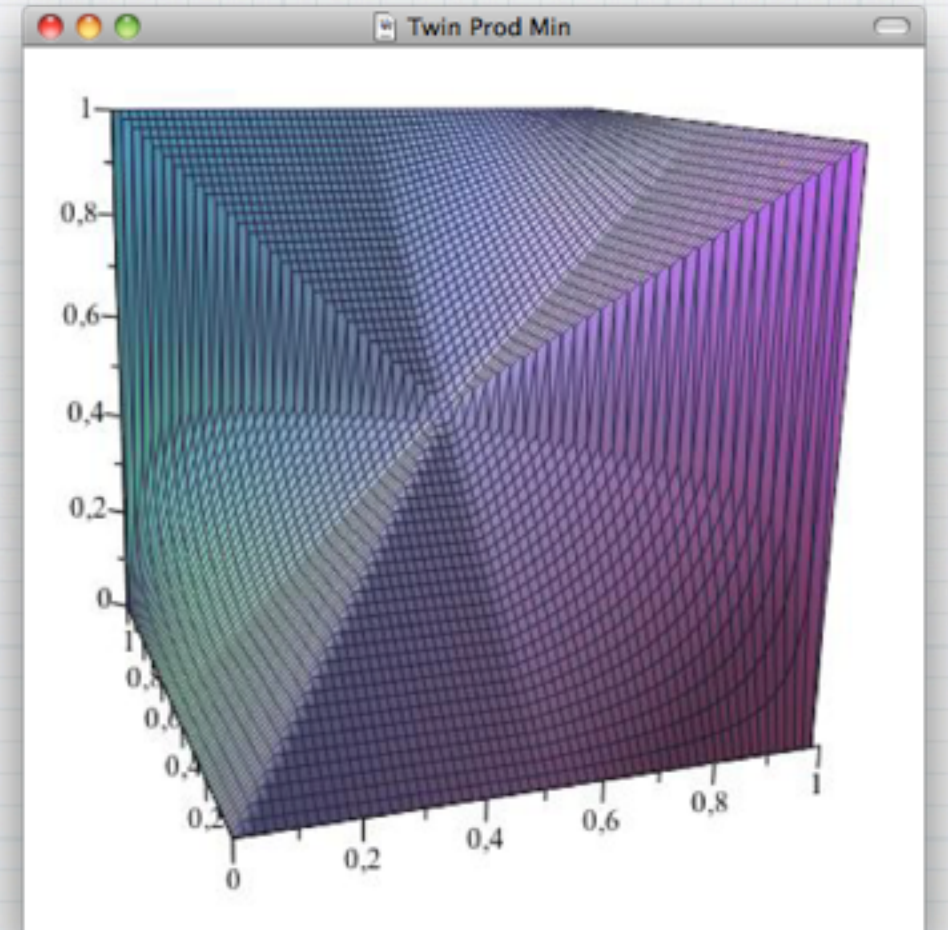

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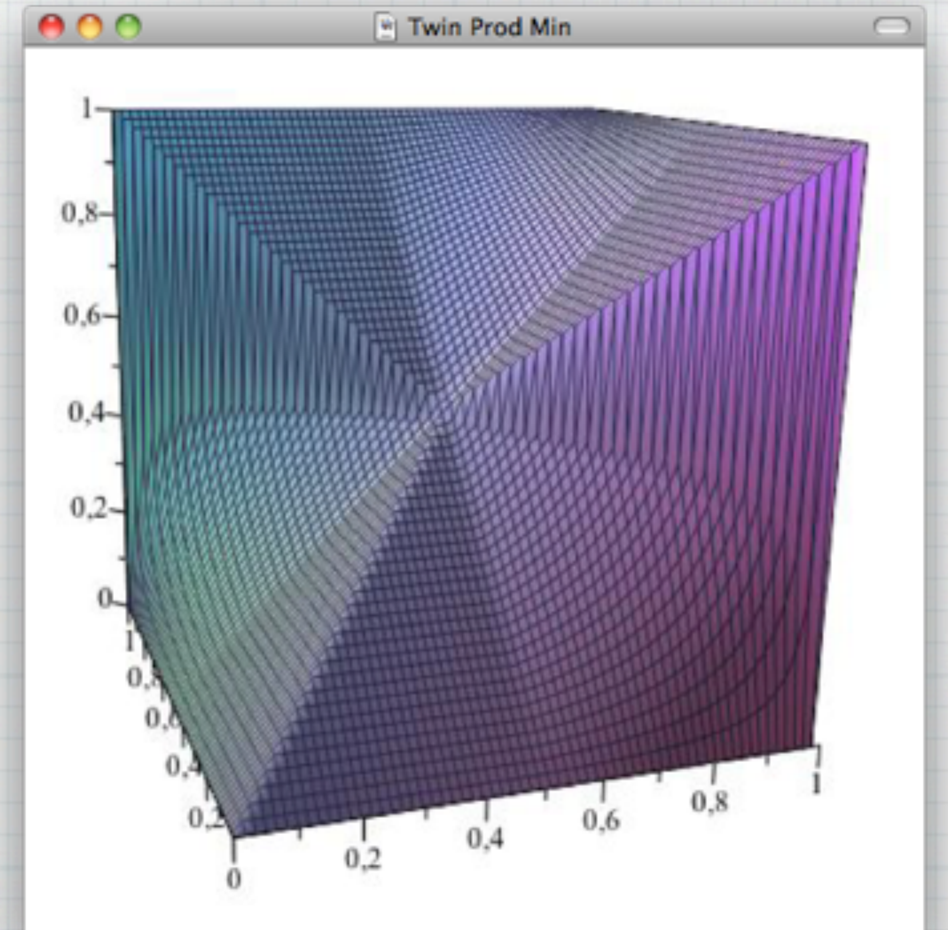

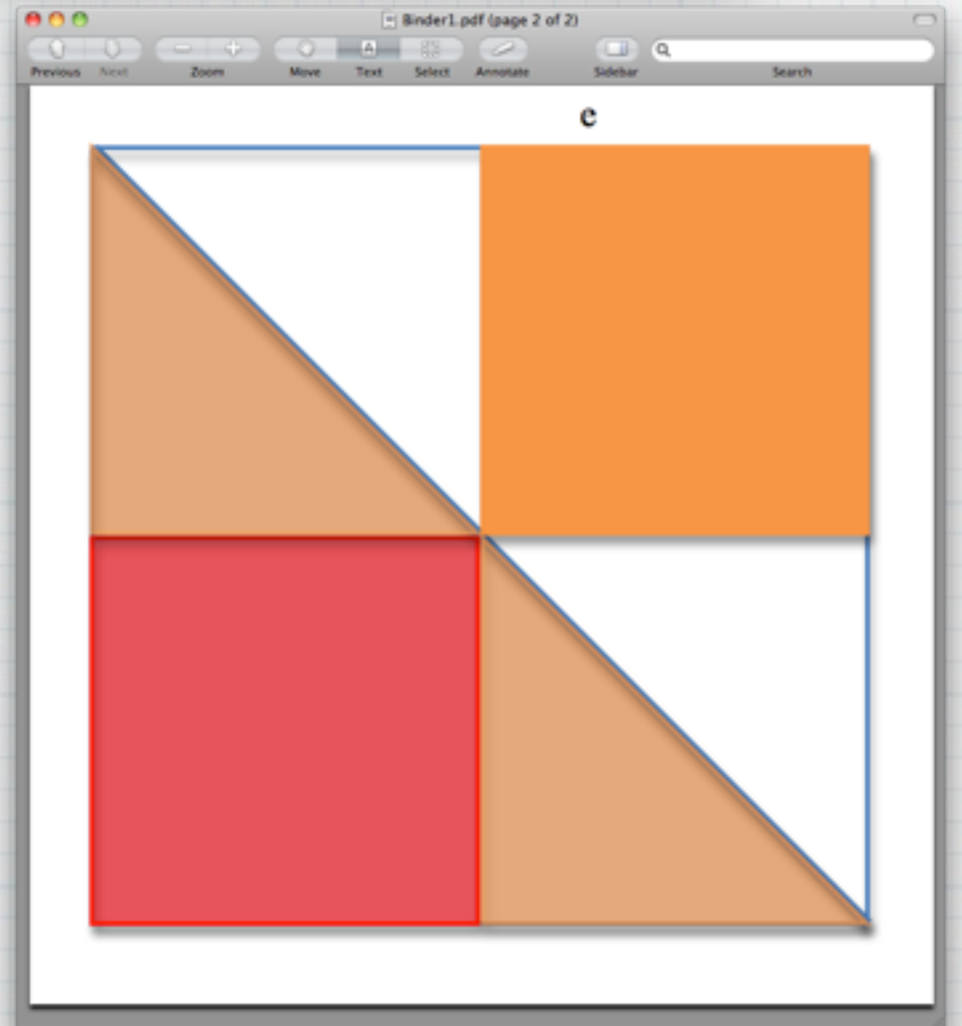

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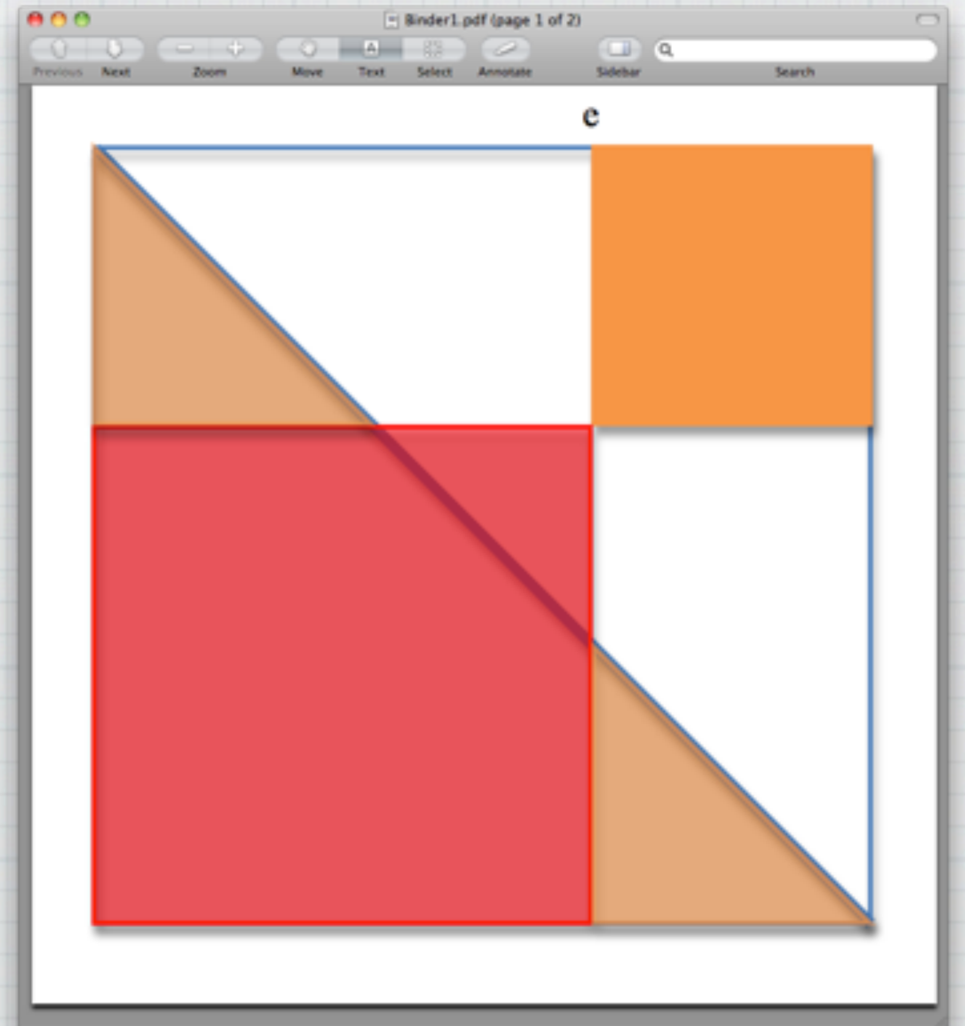

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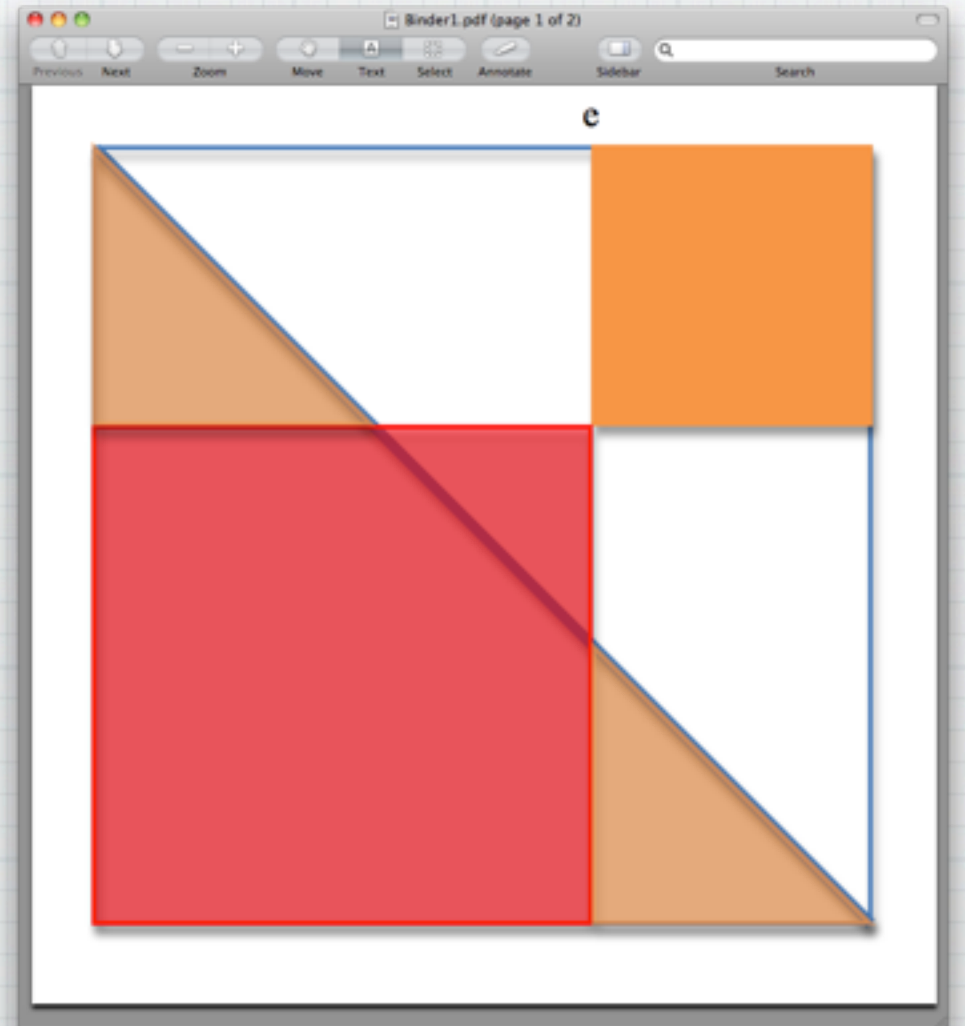

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Twin rotation

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well-defined (1 page)

Proposition 5 *1. $\mathcal{U}_\otimes^\oplus$ in Definition 4 is well-defined,*

2. it is an involutive FL_e-algebra if and only if $\circledast$ is associative,

* THEOREM:
Every conic involutive uninorm is the twin rotation of its underlying t-norm and t-conorm.

Chains

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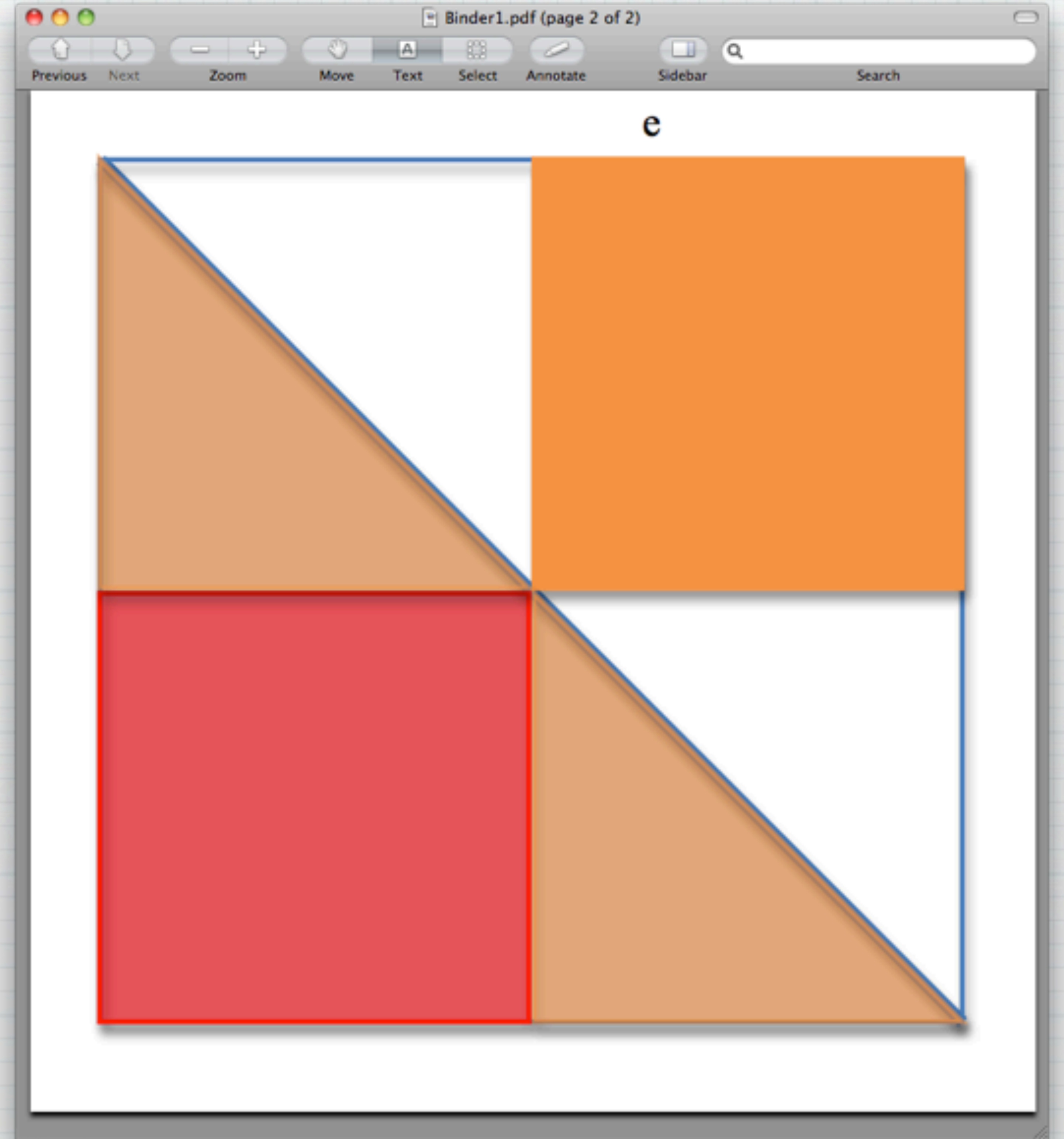
Corollary 2 *Any involutive FL_e-chain is the twin-rotation of its underlying t-norm and t-conorm.*

Complete, densely ordered chains with e=f

* Skew dualization on complete, densely ordered posets
* Classification of involutive FLe-algebras with e=f on [0,1]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Skew dualization on complete, densely ordered chains

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Definition 5 [1] For any commutative residuated lattice on a complete and dense chain $\langle X, \leq, \oplus, \rightarrow_\oplus, 1\rangle$, define $\circledbullet : X \times X \rightarrow X$ by $x \circledbullet y = \inf\{u \oplus v \mid u > x, v > y\}$, and call it the *skewed pair* of $\oplus$.

Definition 6 [1] Let $(L_2, \leq)$ be a complete, dense chain and $L_1 \subseteq L_2$. Let $(L_1, \oplus, \rightarrow_\oplus, \leq, \top)$ be a commutative residuated lattice on a complete and dense chain, and let $'$ be an order reversing involution on L_2. The operation $\odot$ is said to be dual to $\oplus$ with respect to $'$ if $\odot$ is a binary operation on $(L_1)' = \{x' \mid x \in L_1\}$ given by $x \odot y = (x' \oplus y')'$. We say that the operation $\circledbullet$ is *skew dual* to the residuated operation $\oplus$ with respect to $'$ if $\circledbullet$ is the dual of the skewed pair of $\oplus$.

Classification of involutive FLe-algebras with e=f on [0,1]

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Theorem 4 *Any involutive uninorm on* $[0,1]$ *with* $e = f$ *can be represented by (2) where its undelying t-norm and t-conorm are skew-duals.*

Theorem 5 *For any involutive uninorm on* $[0,1]$ *such that its underlying t-norm* $\odot$ *is continuous, one of the following statements is true:*

1. $\odot$ *is order-isomorphic to the product t-norm or*
2. $\odot$ *is order-isomorphic to the minimum t-norm or*
3. $\odot$ *is order-isomorphic to an ordinal sum with summands all being product t-norms.*

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Holds in a more general setting

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Finite Chains

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Corollary 2 *Any involutive FL_e-chain is the twin-rotation of its underlying t-norm and t-conorm.*

Finite Chains

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Corollary 2 *Any involutive FL_e-chain is the twin-rotation of its underlying t-norm and t-conorm.*

Consider a finite involutive FL_e-chain

$$\mathcal{U}_n = \langle \{1, 2, \ldots, n\}, \circledast, \leq, e, f \rangle$$

and denote its underlying t-norm (which acts on $\{1, 2, \ldots, e\}$) and its underlying t-conorm (which acts on $\{e, e+1, \ldots, n\}$) by $\otimes$ and $\oplus$, respectively. We have

$$\mathcal{U}_n = \mathcal{U}_{\otimes}^{\oplus}$$

Finite Chains

$\mathcal{U}_n = \langle \{1, 2, \ldots, n\}, \circledast, \leq, 1, n, e, f \rangle$.

Call $e - f$ the *rank* of $\mathcal{U}_n$ (or the rank of $\circledast$).

We have that $\mathrm{rank}(\mathcal{U}_n)$ *is necessarily even if* n *is odd, and vice versa.*

Finite Chains

* skew dualisation between non-positive rank algebras and positive rank algebras
* positive rank algebras

Skew dualisation

Definition 4

i. Let $\mathcal{U} = \langle \{1, 2, \ldots, n\}, \circledast, \leq, e, f \rangle$ be an involutive FL_e-algebra with $rank(\mathcal{U}) > 0$. Define the following algebra:

$$\mathcal{U}_{\triangledown} = \langle \{1, 2, \ldots, n+1\}, \circ, \leq, f+1, e \rangle,$$

where $\circ$ is the dual of $\diamond$, and $\diamond$ is derived from $\circledast$ by adding $n+1$ as a new annihilator to it. More formally, for $x, y \in \{1, 2, \ldots, n+1\}$ let

$$x \diamond y = \begin{cases} x \circledast y & \text{if } x, y \in \{1, 2, \ldots, n\} \\ n+1 & \text{if } \max(x, y) = n+1 \end{cases}$$

and let $x \circ y = (x' \diamond y')'$, where $'$ is the order-reversing involution of $\{1, 2, \ldots, n\}$.

Skew dualisation

Definition 4

ii. Let $\mathcal{U} = \langle \{1, 2, \ldots, n+1\}, \circledast, \leq, e, f \rangle$ be an involutive FL_e-algebra, and assume $rank(\mathcal{U}) \leq 0$. Define the following algebra:

$$\mathcal{U}_{\Delta} = \langle \{1, 2, \ldots, n\}, \circ, \leq, f, e-1 \rangle,$$

where $\circ$ is the restriction of the dual of $\circledast$ to $\{1, 2, \ldots, n\}$.

and let $x \circ y = (x' \diamond y')'$, where $'$ is the order-reversing involution of $\{1, 2, \ldots, n\}$.

Skew dualisation

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Theorem 2 (finite skew dualization)

i. *For any involutive* FL_e*-algebra* $\mathcal{U}$ *on* $\{1, 2, \ldots, n\}$ *with rank* $k > 0$, $\mathcal{U}_{\nabla}$ *is an involutive* FL_e*-algebra on* $\{1, 2, \ldots, n+1\}$ *with rank* $1 - k$.

ii. *For any involutive* FL_e*-algebra* $\mathcal{U} = \langle\{1, 2, \ldots, n+1\}, \circledast, \leq, e, f\rangle$ *with rank* $k \leq 0$, $\mathcal{U}_{\Delta}$ *is an involutive* FL_e*-algebra on* $\{1, 2, \ldots, n\}$ *with rank* $1 - k$.

Moreover, we have

$$(\mathcal{U}_{\Delta})_{\nabla} = \mathcal{U} \quad \text{and} \quad (\mathcal{U}_{\nabla})^{\nabla} = \mathcal{U}.$$

1-k

Positive rank algebras

Lemma

If $n \geq 3$, $rank(\mathcal{U}_n) \geq 0$ *then* $\mathcal{U}_n$ *has a subalgebra on* $\{2, \ldots, n-1\}$ *iff* $2 \circledast 2 = 2$.

Positive rank algebras

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It is sufficient to treat the $\top\bot$-indecomposable case.

Positive rank algebras

Lemm

If n

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Consider a positive rank finite involutive uninorm $\circledast$ on $\{1, 2, \ldots, n\}$. If $\circledast$ is not $\top\bot$-indecomposable then it has a subalgebra on $\{2, \ldots, n-1\}$ and thus we have $2 \rightarrow_{\circledast} (n-1) \leq n-1$ which implies $2 \circledast n = n$. Therefore, in addition to (18), $n \circledast \{2, 3, \ldots, n\}$ holds too, that is, the Cayley table of $\circledast$ is uniquely determined by the Cayley table of the subalgebra, which is either $\top\bot$-indecomposable or has a subalgebra on $\{3, \ldots, n-2\}$, etc.

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It is

$$1 \circledast \{1, 2, \ldots, n\} = 1 \quad \text{and} \quad n \circledast \{k, k+1, \ldots, n\} = n. \quad (18)$$

Positive rank algebras

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Theorem 2 (rank 0,1) *We have that* $\circledast$ *is the monoidal operation of a finite involutive* FL_e*-chain with rank* 0 *(resp. rank* 1*) iff* n *is odd (resp.* n *is even) and*

$$x \circledast y = \begin{cases} \min(x, y) & \text{if } x \leq y' \\ \max(x, y), & \text{if } x > y' \end{cases} \qquad (4)$$

Positive rank algebras

ASubL 4.pdf (page 12 of 19)

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Corollary 3 *IUL plus $e \leftrightarrow f$ doesn't have the FMP.*

Positive rank algebras

Denote $\odot$ the drastic t-norm on $\{1, 2, \ldots, n\}$ by

$$x \odot y = \begin{cases} 1 & \text{if } x, y < n \\ \min(x, y) & \text{otherwise} \end{cases} . \qquad (19)$$

Theorem 4 (rank 2) *Let $n \geq 3$ odd. We have that $\circledast$ is a $\top\bot$-indecomposable finite involutive uninorm on the chain $\{1, 2, \ldots, n\}$ with rank 2 if and only if its underlying t-norm (resp. t-conorm) is $\odot$ on the $\frac{n+3}{2}$-element chain (resp. an arbitrary t-conorm on the $\frac{n-1}{2}$-element chain).*

Positive rank algebras

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Corollary 4 *Denote C_n the number of conorm operations on an n-element chain. The number of $\top\bot$-indecomposable involutive uninorms on an n-element chain with rank 2 equals to $C_{\frac{n-1}{2}}$. The number of involutive uninorms on an n-element chain with rank 2 equals to $\sum_{i=1}^{\frac{n-1}{2}} C_i$.*

arbitrary t-conorm on the $\frac{n-1}{2}$-element chain).

Positive rank algebras

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Corollary 5 (rank -1) *Let $n \geq 4$ even. We have that $\circ$ is a finite involutive uninorm on the chain $\{1, 2, \ldots, n\}$ with rank -1 satisfying $(n-1) \circ (n-1) = n$ if and only if its underlying t-norm (resp. t-conorm) is an arbitrary t-norm $\otimes$ on the $\frac{n}{2}$-element chain satisfying $2 \otimes 2 = 2$ (resp. the dual of $\odot$ on the $\frac{n+2}{2}$-element chain).*

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n-element chain with rank 2 equals to $\sum_{i=1}^{\frac{n-1}{2}} C_i$.

arbitrary t-conorm on the $\frac{n-1}{2}$-element chain).

Positive rank algebras

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Proposition 4 (rank n-1) *Let $n \geq 1$. We have that $\circledast$ is a finite involutive uninorm on the chain $\{1, 2, \ldots, n\}$ with rank $n-1$ if and only if its underlying t-norm (resp. t-conorm) is any Girard monoid on the n-element chain (resp. the t-conorm on the one-element chain).*

Positive rank algebras

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Proposition 4 (rank n-1) *Let $n \geq 1$. We have that $\circledast$ is a finite involutive uninorm on the chain $\{1, 2, \ldots, n\}$ with rank $n-1$ if and only if its underlying t-norm (resp. t-conorm) is any Girard monoid on the n-element chain (resp. the t-conorm on the one-element chain).*

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Corollary 6 (rank 3-n) *Let $n \geq 2$. We have that $\circledast$ is a finite involutive uninorm on the chain $\{1, 2, \ldots, n\}$ with rank $3-n$ if and only if its underlying t-norm (resp. t-conorm) is the (unique) t-norm, namely, the minimum, on the two-element chain (resp. the dual of any Girard monoid on the $n-1$-element chain).*

Positive rank algebras

Theorem 5 (rank n-3) *Let $n \geq 3$. We have that $\circledast$ is a finite involutive uninorm on the chain $\{1, 2, \ldots, n\}$ with rank $n-3$ if and only if its underlying t-norm satisfies condition 1 in Definition 2 and its underlying t-conorm coincide with the maximum operation on the two-element chain.*

Positive rank algebras

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Theorem 5 (rank n-3) *Let $n \geq 3$. We have that $\circledast$ is a finite involutive uninorm on the chain $\{1, 2, \ldots, n\}$ with rank $n-3$ if and only if its underlying t-norm satisfies condition 1 in Definition 2 and its underlying t-conorm coincide with the maximum operation on the two-element chain.*

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Corollary 7 (rank 5-n) *Let $n \geq 5$. We have that $\circledast$ is a finite involutive uninorm on the chain $\{1, 2, \ldots, n\}$ with rank $5-n$ if and only if its underlying t-conorm coincide with the minimum operation on the three-element chain, and its underlying t-conorm satisfies condition 2 in Definition 2.*

Thank you!