Divisible pseudo-BCK-algebras

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Divisible porims/residuated lattices

- A porim is a structure $(A; \cdot, \to, \rightsquigarrow, 1, \leq)$ such that $(A; \cdot, 1, \leq)$ is an integral pomonoid and

$$x \cdot y \leq z \quad \Leftrightarrow \quad x \leq y \to z \quad \Leftrightarrow \quad y \leq x \rightsquigarrow z$$

  for all $x, y, z \in A$.
- A porim A is called divisible if
  - $x \leq y$ iff $x = y \cdot a = b \cdot y$ for some $a, b \in A$, or
  - A satisfies the identity

$$x \cdot (x \rightsquigarrow y) = (y \to x) \cdot y.$$

- Divisible porims = pseudo-hoops
- Divisible integral residuated lattices = integral GBL-algebras
- Jipsen & Montagna: finite GBL-algebras are commutative

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Pseudo-BCK-algebras

- A pseudo-BCK-algebra [Georgescu & Iorgulescu] is a structure $(A; \to, \rightsquigarrow, 1, \leq)$ such that $\leq$ is a partial order under which 1 is the top element of A, and the following conditions are satisfied (for all $x, y, z \in A$):

$$x \to y \leq (y \to z) \rightsquigarrow (x \to z), \quad x \rightsquigarrow y \leq (y \rightsquigarrow z) \to (x \rightsquigarrow z),$$
$$1 \to x = x, \quad 1 \rightsquigarrow x = x,$$
$$x \leq y \quad \Leftrightarrow \quad x \to y = 1 \quad \Leftrightarrow \quad x \rightsquigarrow y = 1.$$

- $(A; \to, \rightsquigarrow, 1, \leq) \quad \longmapsto \quad (A; \to, \rightsquigarrow, 1)$
- Pseudo-BCK-algebras = the $\{\to, \rightsquigarrow, 1\}$-subreducts of porims/integral residuated lattices

Divisible pseudo-BCK-algebras

- A porim is divisible iff it satisfies

$$(x \to y) \to (x \to z) = (y \to x) \to (y \to z),$$
$$(x \rightsquigarrow y) \rightsquigarrow (x \rightsquigarrow z) = (y \rightsquigarrow x) \rightsquigarrow (y \rightsquigarrow z).$$

- We call a pseudo-BCK-algebra divisible if it satisfies these identities.
- Vetterlein:

$$(x \to y) \to (x \to z) = x \to ((x \rightsquigarrow y) \to z),$$
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n-potent algebras

- A porim is n-potent ($n \in \mathbb{N}$) if $x^n = x^{n+1}$.
- Pseudo-BCK-algebras:
  - Notation: $x^n \to y = x \to (\ldots \to (x \to y) \ldots)$
  - We call a pseudo-BCK-algebra n-potent if for all x, y,

$$x^n \to y = 1 \quad \text{iff} \quad x^{n+1} \to y = 1.$$

  - Equivalently:

$$x^n \to y = x^{n+1} \to y \quad \text{or} \quad x^n \rightsquigarrow y = x^{n+1} \rightsquigarrow y$$

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Deductive systems

Let $(A; \to, \rightsquigarrow, 1)$ be a pseudo-BCK-algebra. A deductive system is $X \subseteq A$ such that

- $1 \in X$,
- if $a \in X$ and $a \to b \in X$ (or $a \rightsquigarrow b \in X$), then $b \in X$.

A deductive system is normal if for all $a, b \in A$,

- $a \to b \in X$ iff $a \rightsquigarrow b \in X$, or
- if $a \in X$, then $(a \to b) \to b \in X$ and $(a \rightsquigarrow b) \rightsquigarrow b \in X$.

If X is a normal d.s., then $\theta_X = \{\langle a, b \rangle \mid a \to b, b \to a \in X\}$ is a congruence such that $A/X = A/\theta_X$ is a pseudo-BCK-algebra.

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Subdirectly irreducible normal divisible pseudo-BCK-algebras

Blok & Ferreirim: hoops
Jipsen & Montagna: integral GBL-algebras

Ordinal sums

Let $(I;\leq)$ be a linearly ordered set and $\{A_i \mid i \in I\}$ be a family of pseudo-BCK-algebras such that $A_i \cap A_j = 1$ for all $i \neq j$. The ordinal sum of the algebras $(A_i; \to_i, \rightsquigarrow_i, 1)$ is the pseudo-BCK-algebra $\bigoplus_{i\in I} A_i = (\bigcup_{i\in I} A_i; \to, \rightsquigarrow, 1)$ where

$$x \to y = \begin{cases} x \to_i y & \text{if } x, y \in A_i \text{ for some } i, \\ 1 & \text{if } x \in A_i \setminus \{1\} \text{ and } y \in A_j \text{ for some } i < j, \\ y & \text{if } x \in A_i \text{ and } y \in A_j \setminus \{1\} \text{ for some } i > j, \end{cases}$$

and $\rightsquigarrow$ is defined in the same way.

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Subdirectly irreducible normal divisible pseudo-BCK-algebras

Cone algebras [Bosbach]

- Let $(G; \cdot, {}^{-1}, 1, \leq)$ be a lattice-ordered groups. Then every subalgebra of the algebra $(G^{-}; \rightarrow, \rightsquigarrow, 1)$ where

$$x \rightarrow y = yx^{-1} \wedge 1 \quad \text{and} \quad x \rightsquigarrow y = x^{-1}y \wedge 1$$

is a cone algebra.

- A cone algebra is a divisible pseudo-BCK-algebra satisfying the identity

$$(x \rightarrow y) \rightsquigarrow y = (y \rightsquigarrow x) \rightarrow x.$$

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Subdirectly irreducible normal divisible pseudo-BCK-algebras

Theorem

A non-trivial normal divisible pseudo-BCK-algebra A is subdirectly irreducible iff it is of the form $A = B \oplus C$ where C is a non-trivial subdirectly irreducible linearly ordered cone algebra.

Theorem

Every n-potent divisible pseudo-BCK-algebra is a BCK-algebra.
Every finite divisible pseudo-BCK-algebra is a BCK-algebra.

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Poset products

Jipsen & Montagna:
Let $(I; \leq)$ be a poset and let $\{A_i \mid i \in I\}$ be a family of MV-chains (with the same 0 and 1). Let $A = \bigotimes_{i \in I} A_i$ be the subset of $\prod_{i \in I} A_i$ defined as follows:

$$a \in A \quad \text{iff} \quad \text{whenever } a(i) \neq 1, \text{then } a(j) = 0 \text{ for all } j < i.$$

If the multiplication and the lattice operations are defined pointwise, then $\bigotimes_{i \in I} A_i$ is a GBL-algebra where

$$(a \to b)(i) = \begin{cases} a(i) \to b(i) & \text{if } a(j) \leq b(j) \text{ for all } j < i, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem

Every finite divisible BCK-algebra embeds into a poset product of linearly ordered MV-algebras.

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Let Γ be the set of all completely meet-irreducible deductive systems $M \neq A$, ordered by inclusion. Then for every $M \in \Gamma$, A/M is subdirectly irreducible, so $A/M = B_M \oplus C_M$ where C_M is a finite MV-chain. Then A embeds into the poset product $\bigotimes_{M \in \Gamma} C_M$.

THANK YOU!