On o-minimal MV-chains

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Algebraic Semantics for Uncertainty and Vagueness

Salerno, May 2011

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- A set of n-tuples $A \subseteq M^n$ is said to be parametrically definable if there is some L-formula $\phi(x_1, \ldots, x_n, y_1, \ldots, y_k)$ and $b_1, \ldots, b_k \in M$ so that $A = \{(a_1, \ldots, a_n) : \mathbf{M} \models \phi(a_1, \ldots, a_n, b_1, \ldots, b_k)\}$.

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- A linearly ordered structure $\mathbf{M}$ is said to be *o-minimal* if any parametrically definable subset of M is a finite union of intervals in M.
- A first-order theory T is said to be *o-minimal* if every model of T is o-minimal.

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(*iii*) a real closed field in the language $L = \langle +, \cdot, 0, 1, < \rangle$.

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- Any o-minimal ordered abelian group is also divisible [Pillay, Steinhorn (1986)].
- Indeed, the only definable non-trivial subgroup of an o-minimal ordered abelian group **G** is **G** itself.
- Moreover, in any o-minimal ordered abelian group it is possible to define a divisible group.
- We have that an ordered abelian group is divisible IFF it is o-minimal IFF its theory admits QE [Pillay, Steinhorn (1986); Lenski (1989)].

O-minimality and Quantifier Elimination

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- The theory of the ordered group of integers in the language $\langle +, -, 0, <, \{P_n\} \rangle$, with $n = 2, 3, \ldots$ has QE but is not o-minimal. Indeed, the formula $\exists y\ 2y = x$ defines an infinite union of intervals.

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- The theory of the ordered group of integers in the language $\langle +, -, 0, <, \{P_n\} \rangle$, with $n = 2, 3, \ldots$ has QE but is not o-minimal. Indeed, the formula $\exists y \; 2y = x$ defines an infinite union of intervals.
- The theory of BL-chains representable as an infinite ordinal sum $\bigoplus_{i \in I} \mathbf{A}_i$ of divisible MV-chains, with a densely ordered index set I with a minimum and without a maximum has QE in the language of BL-chains. However, the set of idempotents is definable [Cortonesi, M., Montagna (2010)].

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- Can we exploit the characterization for ordered Abelian groups?
- Every divisible MV-chain is o-minimal...
- but not every o-minimal MV-chain is divisible.

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- By an *unnested atomic formula* in L we mean one of the following formulas:

(i)	$x = y$,	$(x < y)$;	
(ii)	$x = c$,	$(x < c)$,	for some constant symbol $c \in \mathrm{L}$;
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- A formula is called unnested if all its atomic subformulas are unnested.
- For a first-order language $\mathrm{L} = \langle <, f_1, \dots, f_n, c_1, \dots, c_m \rangle$, every formula is equivalent to an unnested formula.

Interpretation (II)

- Let T_1 and T_2 be two theories in the the languages $\mathrm{L}_1 = \langle <, f_1, \ldots, f_n, c_1, \ldots, c_m \rangle$ and $\mathrm{L}_2 = \langle <, f'_1, \ldots, f'_{n'}, c'_1, \ldots, c'_{m'} \rangle$, respectively.

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- T_1 is *interpretable* into T_2 (with parameters) if there exists an L_2-formula $\chi(z)$, and for every $\mathbf{M}_1 \models \mathrm{T}_1$ there exists a $\mathbf{M}_2 \models \mathrm{T}_2$ (unique up to isomorphism) such that:

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  (*i*) there exists a bijection $h_{M_1} : M_1 \to \{a \mid \mathbf{M}_2 \models \chi(a)\}$ from the domain of $\mathbf{M}_1$ into the set defined by the L_2-formula $\chi(z)$ over the domain of $\mathbf{M}_2$;

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- Let T_1 and T_2 be two theories in the languages L_1 and L_2, respectively. Suppose that T_1 is interpretable in T_2. Then for every $\mathbf{M}_1 \models \mathrm{T}_1$ and for each L_1-formula $\varphi(\overline{x})$, there exists an L_2-formula $\varphi^\sharp(\overline{x})$ so that, for all $\overline{b} \in M_1$

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O-minimality and Interpretation

Theorem

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Corollary

The theory of divisible MV-chain is interpretable in the theory of ordered divisible abelian groups, therefore it is o-minimal.

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Are all o-minimal MV-chains divisible?

Perfect MV-chains (I)

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- If such an n exists, then $ord(x) = n$, while if it does not exists, $ord(x) = \infty$.
- An MV algebra is called *perfect* if for every $x \neq 0$, $ord(x) = \infty$ if and only if $ord(\neg(x)) < \infty$.

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- Every perfect MV chain $\mathbf{A} \cong \Gamma(\mathbb{Z} \times \mathbf{G}, (1,0))$ [Di Nola, Lettieri (1994)].

Perfect MV-chains (III)

- An MV-algebra is perfect IFF it satisfies the sentence [Belluce, Di Nola, Gerla (2007)]

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- Semidivisible perfect MV-chains are exactly those chains **A** such that $\mathbf{A} \cong \Gamma(\mathbb{Z} \times \mathbf{G}, (1, 0))$, where **G** is a divisible ordered abelian group.

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Every Perfect MV-chain is semidivisible IFF it is o-minimal.

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- An MV algebra $\mathbf{A}$ has rank n iff $\mathbf{A}/Rad(\mathbf{A}) \cong \mathbf{S}_n$
- A local MV-algebra $\mathbf{A}$ of rank n is *radical retractive* if $\mathbf{A}/Rad(\mathbf{A})$ is a subalgebra of $\mathbf{A}$.

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- Every radical retractive local MV-chain of rank n $\mathbf{A} \cong \Gamma(\mathbb{Z} \times \mathbf{G}, (n,0))$ [Di Nola, Esposito, Gerla (2007)].

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- Semidivisible radical retractive local MV-chain of rank n are exactly those chains $\mathbf{A}$ such that $\mathbf{A} \cong \Gamma(\mathbb{Z} \times \mathbf{G}, (n, 0))$, where $\mathbf{G}$ is a divisible ordered abelian group.

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- Items of the form $\overline{a}/\phi$, where ϕ is an equivalence formula and $\overline{a}$ a tuple, are known as *imaginary elements* of $\mathbf{A}$.
- L-structure $\mathbf{A}$ has *elimination of imaginaries* if for every equivalence formula $\theta(\overline{x}, \overline{y})$ of $\mathbf{A}$ and each tuple $\overline{a}$ in $\mathbf{A}$ there is a formula $\phi(\overline{x}, \overline{y})$ of L such that the equivalence class $\overline{a}/\theta$ of $\overline{a}$ can be written as $\phi(A^n, \overline{b})$ for some unique tuple $\overline{b}$ from $\mathbf{A}$.

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- An equivalence formula of **A** is a formula $\phi(\overline{x}, \overline{y})$ of L, without parameters, such that the relation $\{(a, b) : \mathbf{A} \models \phi(\overline{a}, \overline{b})\}$ is a non-empty equivalence relation E_ϕ.
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- L-structure **A** has *elimination of imaginaries* if for every equivalence formula $\theta(\overline{x}, \overline{y})$ of **A** and each tuple $\overline{a}$ in **A** there is a formula $\phi(\overline{x}, \overline{y})$ of L such that the equivalence class $\overline{a}/\theta$ of $\overline{a}$ can be written as $\phi(A^n, \overline{b})$ for some unique tuple $\overline{b}$ from **A**.
- We say that **A** has *uniform elimination of imaginaries* if the same holds, except that ϕ depends only on θ and not on $\overline{a}$.

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- Every o-minimal structure with definable Skolem functions and at least two constant elements has uniform elimination of imaginaries [Hodges (1993)].
- Recall that a theory T has definable Skolem functions if for every formula $\phi(\overline{x}, y)$, with $\overline{x}$ not empty, there is a formula $\psi(\overline{x}, y)$ such that

$$\mathrm{T} \vdash \forall \overline{x}\ (\exists y\ \phi(\overline{x}, y) \to (\exists_{=1} y\ \psi(\overline{x}, y) \wedge \forall y\ (\psi(\overline{x}, y) \to \phi(\overline{x}, y)))).$$

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  1. the theory of divisible MV-chains has quantifier elimination;
  2. every MV-chain **A** can be embedded into a divisible one **B** such that for every $b \in B$ there is a formula $\phi(x)$ (with parameters from A) such that $\mathbf{B} \models \phi(b)$ and $\mathbf{B} \models (\exists^{\leq n} x)\phi(x)$ for some n;

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  3. every MV-chain **A** can be embedded into a divisible one **B** such that there is no automorphism of **B** fixing **A** other than the identity.

THANKS!