# The Euler characteristic of a (monodimensional) polyhedron as a valuation on a vector lattice

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Algebraic Semantics for Uncertainty and Vagueness
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# Polyhedra

Let $x_0, \dots, x_n \in \mathbb{R}^m$ be affinely independent points
(i.e. $x_1 - x_0, \dots, x_n - x_0$ linearly independent)

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$$\sigma_n = (x_0, \dots, x_n) = \left\{ \sum_{i=0}^{n} \lambda_i x_i \ : \ \lambda_i \in \mathbb{R},\ \lambda_i \geq 0,\ \sum_{i=0}^{n} \lambda_i = 1 \right\}$$

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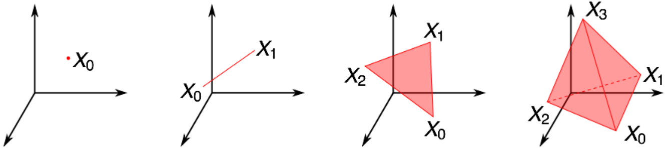

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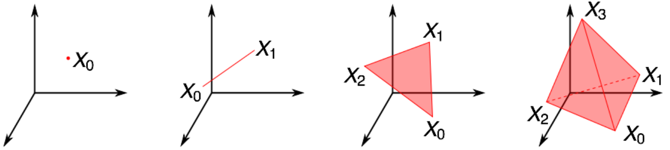


A face of $\sigma_n$ is any $\tau_p = (x_{i_0}, \ldots, x_{i_p})$, $\{x_{i_0}, \ldots, x_{i_p}\} \subseteq \{x_0, \ldots, x_n\}$

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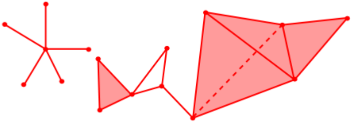

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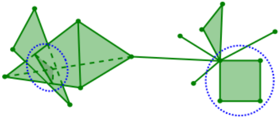

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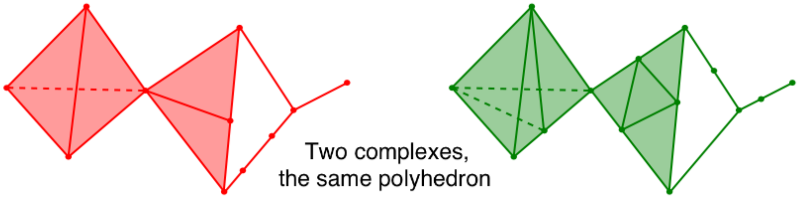

# The Euler-Poincaré characteristic

Let $K$ be a triangulation of the polyhedron $P$,
the Euler-Poincaré characteristic of $P$ is the number

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where, for all $n$, $\alpha_n$ is the number of $n$-simplices of $K$.

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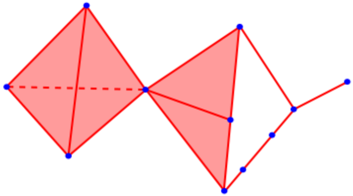

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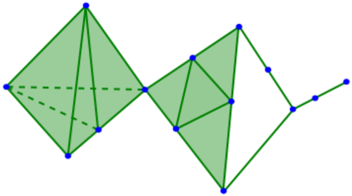

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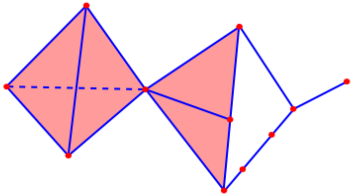
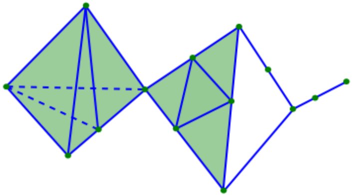


11 − 16



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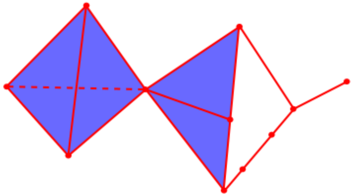

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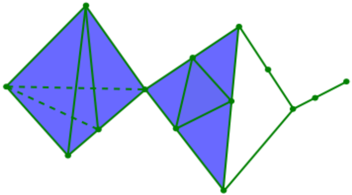

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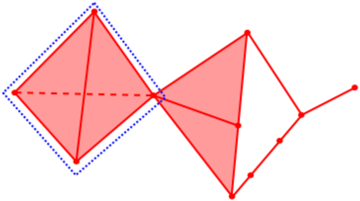
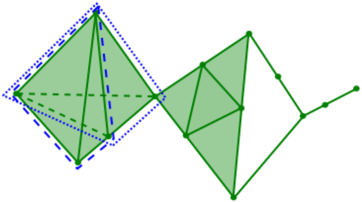


$11 - 16 + 6 - 1 = 0$



$14 - 22 + 10 - 2 = 0$

# Vector lattices

A (real) vector lattice is an algebra $\mathbf{V} = (V, +, \wedge, \vee, \{\lambda\}_{\lambda \in \mathbb{R}}, 0)$ such that

- $(V, +, \{\lambda\}_{\lambda \in \mathbb{R}}, 0)$ is a vector space,
- $(V, \wedge, \vee)$ is a lattice,
- for all $t, v, w \in V$, $\quad t + (v \wedge w) = (t + v) \wedge (t + w)$,
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A strong unit is an element $u \in V$ such that for all $0 \leq v \in V$ there exists a $0 \leq \lambda \in \mathbb{R}$ such that $v \leq \lambda u$.
A unital vector lattice is a pair $(\mathbf{V}, u)$, where $\mathbf{V}$ is a vector lattice and $u$ is a strong unit of $\mathbf{V}$.

# Vector lattices

A function $f : \mathbb{R}^m \to \mathbb{R}^n$ is piecewise linear if there are finitely many linear polynomials $w_1, \ldots, w_s$ such that

$$\forall x \in \mathbb{R}^m \; \exists i \in \{1, \ldots, s\} \; : \; f(x) = w_i(x).$$

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Let $P$ a polyhedron in $\mathbb{R}^m$.

$$F(P) = \{f : P \to \mathbb{R} \text{ continuous and piecewise linear}\}$$

$$\nabla(P) = (F(P), +, \min, \max, \{\lambda\}_{\lambda \in \mathbb{R}}, 0)$$

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Baker-Beynon duality: each finitely presented $(\mathbf{V}, u)$ is isomorphic to $(\nabla(P), 1)$, for some $P$ in some Euclidean space $\mathbb{R}^m$.

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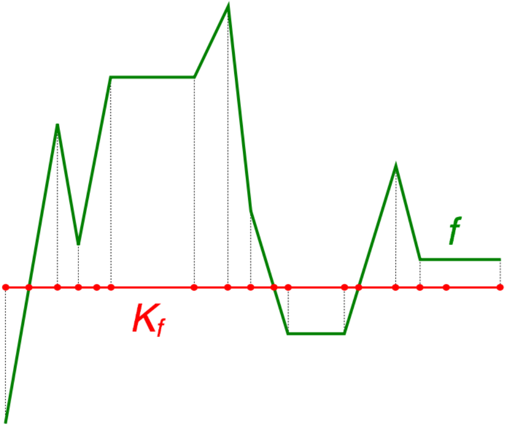

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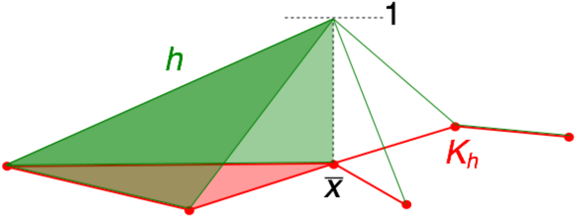
A vl-Schauder hat is an $h \in \nabla(P)$ such that there is a triangulation $K_h$ of $P$ linearizing $h$ and a 0-simplex $\bar{x}$ of $K_h$ such that $h(\bar{x}) = 1$ and $h(x) = 0$ for any other 0-simplices $x$ of $K_h$.



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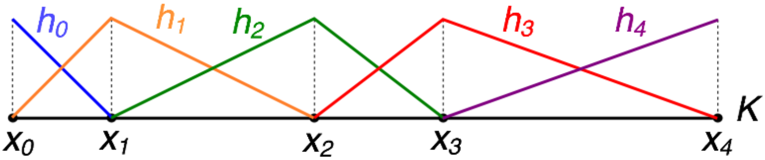

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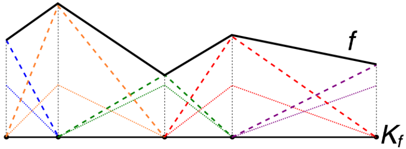
Each $f \in \nabla(P)$ can be seen as a sum $\sum_{i=0}^{n} a_i h_i$ (where $a_i \in \mathbb{R}$) of the vl-Schauder hats $h_0, \ldots, h_n$ of a triangulation $K_f$ linearizing $f$.



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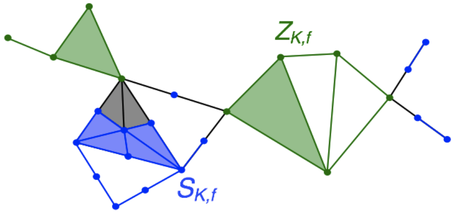

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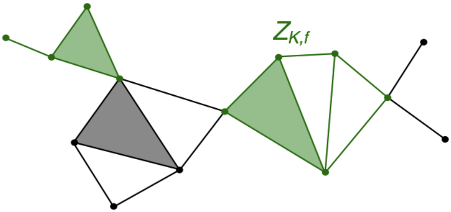

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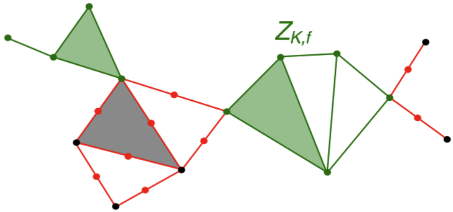

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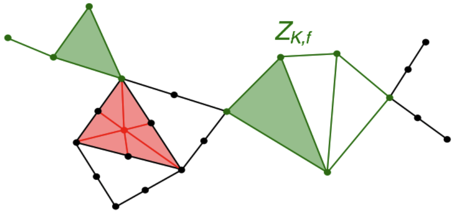

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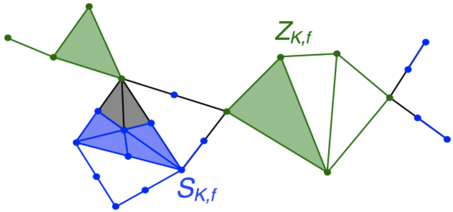

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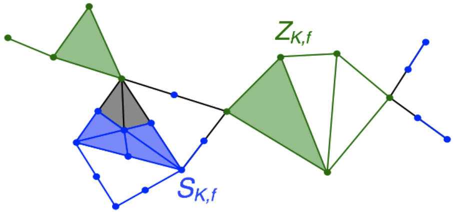


The Euler-Poincaré characteristic of $f$:

$$\chi(f) = \chi(supp(f)) = \chi(S_{K,f})$$

(it does not depend on the choice of $K$).

# Valuations

A vl-valuation on $(\nabla(P), 1)$ is a function $\nu : \nabla(P) \to \mathbb{R}$ such that:

- $\nu(0) = 0$,
- for all $f, g \in \nabla(P)$, $\nu(f \vee g) = \nu(f) + \nu(g) - \nu(f \wedge g)$,
- for all $0 \leq f, g \in \nabla(P)$, $\nu(f + g) = \nu(f \vee g)$,
- for all $0 \leq f, g \in \nabla(P)$, if $f \wedge g = 0$ then $\nu(f - g) = \nu(f) - \nu(g)$.

# Characterization Theorem

**Theorem**

Let $P$ be a polyhedron in $\mathbb{R}^m$, for some integer $m \geq 1$, and let $(\nabla(P), 1)$ be the finitely presented unital vector lattice of real-valued piecewise linear functions on $P$.

Then Euler-Poincaré characteristic is the unique vl-valuation

$$\chi \,:\, \nabla(P) \to \mathbb{R}$$

that assigns the value 1 to each vl-Schauder hat in $\nabla(P)$.

Moreover, the number $\chi(1)$ is the Euler-Poincaré characteristic of the polyhedron $P$.

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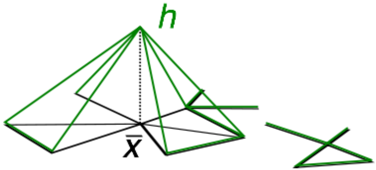

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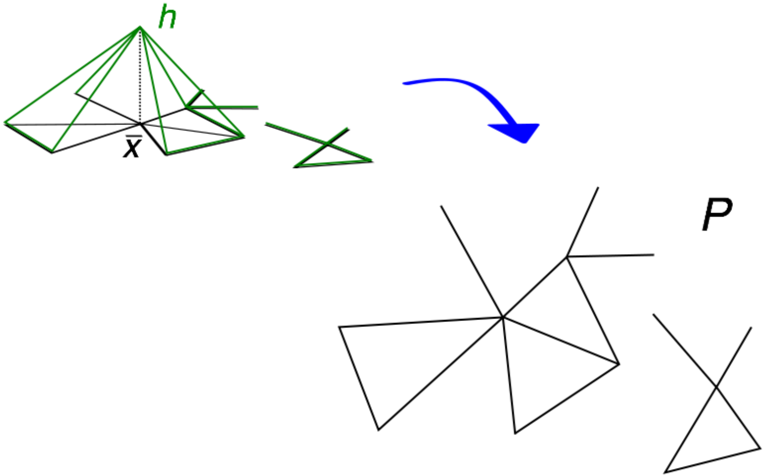

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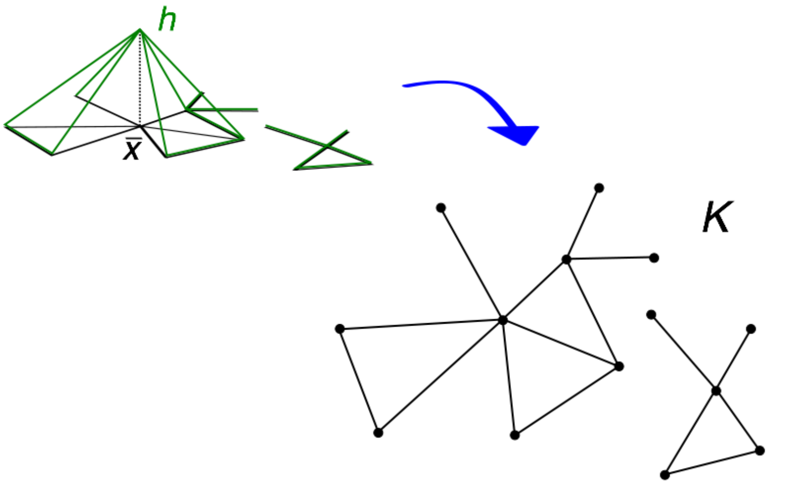

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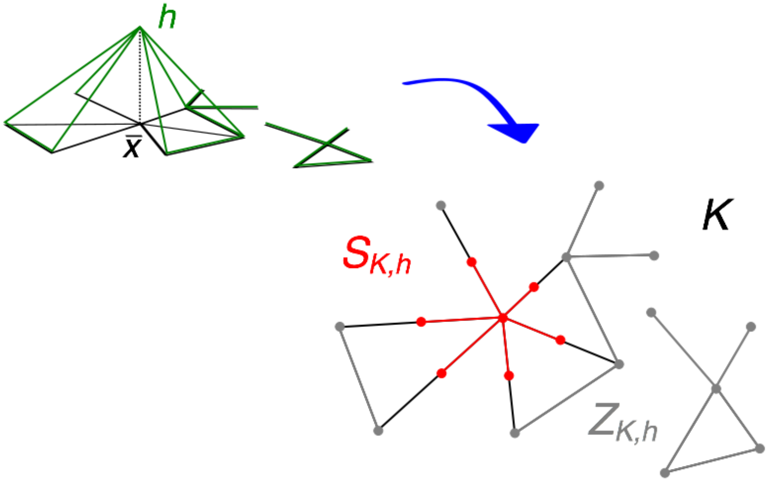

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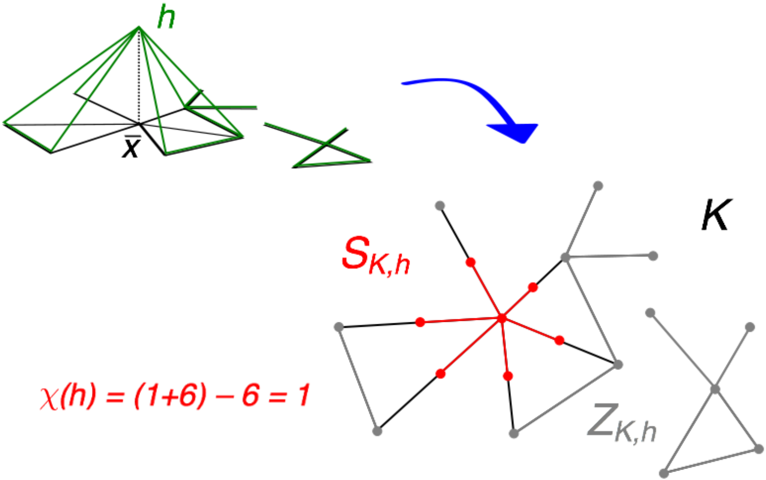

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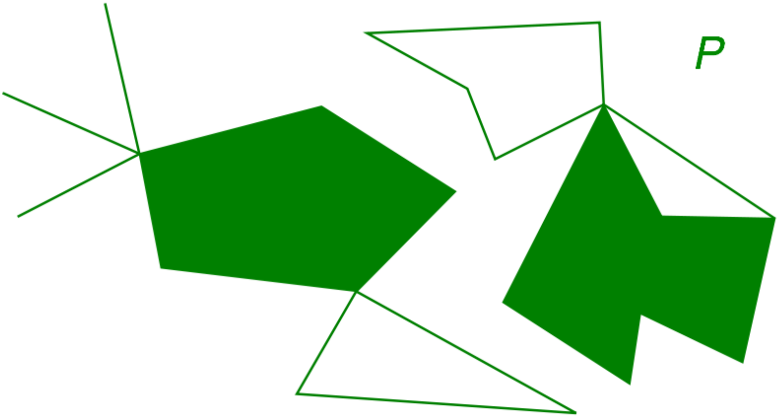
# $\chi(1) = \chi(P)$



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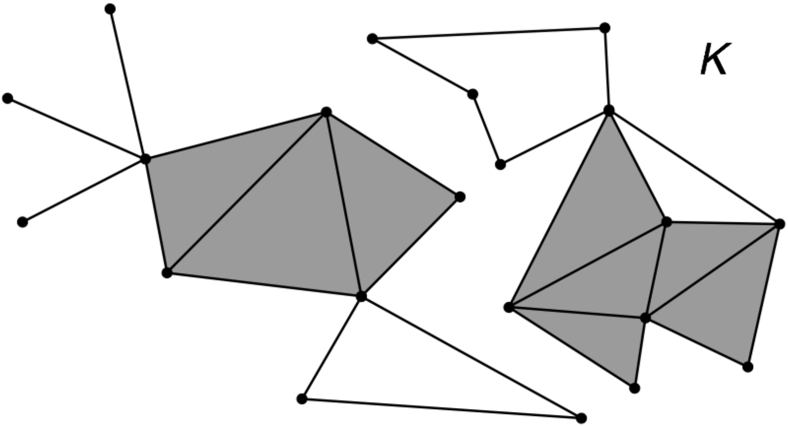

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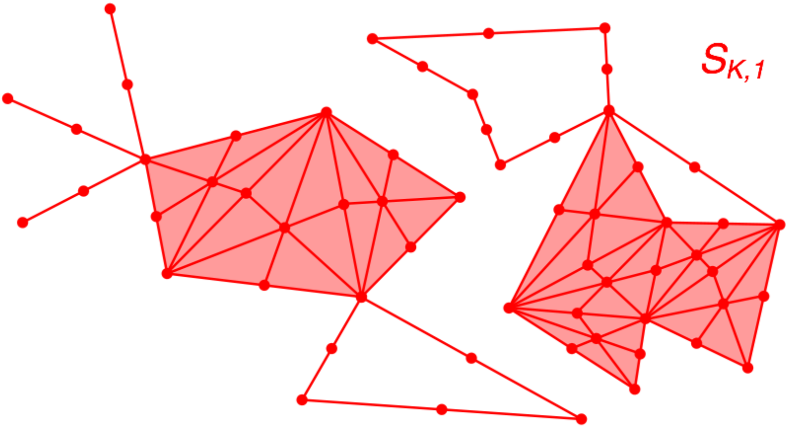

# Characterization Theorem

**Theorem**

Let $P$ be a polyhedron in $\mathbb{R}^m$, for some integer $m \geq 1$, and let $(\nabla(P), 1)$ be the finitely presented unital vector lattice of real-valued piecewise linear functions on $P$.
Then Euler-Poincaré characteristic is the unique vl-valuation

$$\chi \; : \; \nabla(P) \to \mathbb{R}$$

that assigns the value 1 to each vl-Schauder hat in $\nabla(P)$.

Moreover, the number $\chi(1)$ is the Euler-Poincaré characteristic of the polyhedron $P$.

Thank you for your attention.