Kripke fuzzy semantics and algebraic semantics for bi-modal Gödel Logics

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Outline

Motivations (I)

We are looking for models of reasoning with imperfect information which try to address and formalize two different central notions:

- *Truthlikeness / Similarity*: **Closeness to truth** formalized by modal structures, $\mathcal{M} = \langle W, S \rangle$ where W is a set of situations and S is a fuzzy similarity relation, $S : W \times W \mapsto [0,1]$.
  $w \models^{\alpha} \varphi$ if $\exists w' : \ w' \models \varphi$ and $S(w,w') \geq \alpha$
- *Fuzziness / Gradualíty*: **Degrees of truth** formalized by (truth-functional) many-valued models, $v : \mathcal{L} \mapsto [0,1]$.
  $v(\varphi), v(\psi) \in [0,1]$; $v(\varphi \to \psi) = 1$ if $v(\varphi) \leq v(\psi)$

Motivations (II)

Original motivation of our research was to formally characterize the logic used in fuzzy similarity reasoning.

In this sense, we want to deal with assertions like to "*John is approximately tall*", with the intended meaning that the fuzzy proposition "*John is tall*" is "*close to be true*".

Technically, we need to combine elements of many-valued logics (to model fuzziness) and of modal logics (to model the notion of similarity).

$\Rightarrow$ **many-valued modal logics**

Antecedents

We recognize three inspiration sources:

- Ruspini formalized the similarity reasoning using a modal approach over $\{0,1\}$-interpretations. We extend his work by considering $[0,1]$-Gödel interpretations.
- Fitting considered a semantic very close to ours, but his logic is finitely valued and includes finitely many truth constants which are the syntactical counterpart of truth values.
- In the intuitionistic context, there is a lot of examples of modal logic based on intuitionistic logic. For example, the system IK introduced by Fischer-Servi as the natural intuitionistic counterpart of classical modal logic

Semantic (I)

Definition

A Gödel-Kripke model (GK-model) *will be a structure* $M = \langle W, S, e\rangle$ *where* W *is a non-empty set of objects that we call* worlds of M, *and* $S : W \times W \to [0, 1]$, $e : W \times Var \to [0, 1]$ *are arbitrary functions. The pair* $\langle W, S\rangle$ *will be called a* GK-frame.

The many valued Kripke interpretation of bi-modal logic utilized in our work was proposed originally by Fitting with a complete Heyting algebra as algebra of truth values, and he gave a complete axiomatization assuming the algebra was finite and the language had constants for all the truth values.

Semantic (II)

The function $e : W \times Var \to [0,1]$ associates to each world x a valuation $e(x,-) : Var \to [0,1]$ which extends to $e(x,-) : \mathcal{L}_{\Box\Diamond}(Var) \to [0,1]$ by defining inductively on the construction of the formulas:

$e(x,\bot) := 0$
$e(x,\varphi \wedge \psi) := e(x,\varphi) \cdot e(x,\psi)$
$e(x,\varphi \vee \psi) := e(x,\varphi) \curlyvee e(x,\psi)$
$e(x,\varphi \to \psi) := e(x,\varphi) \Rightarrow e(x,\psi)$
$e(x,\bot) := 0$
$e(x,\Box\varphi) := \inf_{y\in W}\{Sxy \Rightarrow e(y,\varphi)\}$
$e(x,\Diamond\varphi) := \sup_{y\in W}\{Sxy \cdot e(y,\varphi)\}.$

Axiomatic

Definition

$\mathcal{G}_{\Box\Diamond}$ is the deductive calculus obtained by adding to $\mathcal{G}$ the schemes

- K$_\Box$ $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$.
- K$_\Diamond$ $\Diamond(\varphi \vee \psi) \to (\Diamond\varphi \vee \Diamond\psi)$.
- F$_\Diamond$ $\neg\Diamond\bot$.
- FS1 $\Diamond(\varphi \to \psi) \to (\Box\varphi \to \Diamond\psi)$.
- FS2 $(\Diamond\varphi \to \Box\psi) \to \Box(\varphi \to \psi)$.

and the inference rules:

- NR$_\Box$ *From* φ *infer* $\Box\varphi$.
- RN$_\Diamond$ *From* $\varphi \to \psi$ *infer* $\Diamond\varphi \to \Diamond\psi$.

Soundness

It is proved in the usual way.

Lemma

$T, \psi \vdash_{\mathcal{G}_{\Box\Diamond}} \varphi$ *implies* $T \vdash_{\mathcal{G}_{\Box\Diamond}} \psi \to \varphi$.

Remark. Changing the algebra $[0,1]$ to a complete Heyting algebra H in the above definitions we have Kripke models valued in a H (HK-models) and the corresponding notion of HK-validity. Then all laws of the intermediate logic determined by H are HK-valid.

Remark. $\mathcal{G}_{\Box\Diamond}$ may be seen deductively equivalent to well known Fischer-Servi system IK plus the prelinearity axiom.

Completeness

We use two principal results:

Lemma

Let $Th\mathcal{G}_{\Box\Diamond}$ be the set of theorems of $\mathcal{G}_{\Box\Diamond}$ with no assumptions, then for any theory T and formula φ in $\mathcal{L}_{\Box\Diamond}$: $T \vdash_{\mathcal{G}_{\Box\Diamond}} \varphi$ if and only if $T \cup Th\mathcal{G}_{\Box\Diamond} \vdash_{\mathcal{G}} \varphi$.

Theorem

i) If $T \cup \{\varphi\} \subseteq \mathcal{L}(X)$, then $T \vdash_{\mathcal{G}} \varphi$ implies $\inf v(T) \leq v(\varphi)$ for any valuation $v : X \to [0,1]$. ii) If T is countable, and $T \nvdash_{\mathcal{G}} \varphi_{i_1} \vee .. \vee \varphi_{i_1}$ for each finite subset of a countable family $\{\varphi_i\}_i$ there is a valuation $v : L \to [0,1]$ such that $v(\theta) = 1$ for all $\alpha \in T$ and $v(\varphi_i) < 1$ for all i.

Canonical Model

We define for each finite *fragment* $F \subseteq \mathcal{L}_{\Box\Diamond}$ a canonical model $M_F = (W, S^F, e^F)$ is defined as follows.

- W: is the set of valuations $v : Var \cup X \to [0,1]$ such that $v(Th\mathcal{G}_{\Box\Diamond}) = 1$ when $Th\mathcal{G}_{\Box\Diamond}$ is considered as a subset of $\mathcal{L}(Var \cup X)$.
- S^F: $S^F vw = \inf_{\psi \in F}\{(v(\Box\psi) \to w(\psi)) \cdot (w(\psi) \to v(\Diamond\psi))\}$.
- e^F: $e^F(v,p) = v(p)$ for any $p \in Var$.

where $X := \Box\mathcal{L}_{\Box\Diamond} \cup \Diamond\mathcal{L}_{\Box\Diamond}$, with $\Box\mathcal{L}_{\Box\Diamond}$ and $\Diamond\mathcal{L}_{\Box\Diamond}$ denoting the sets of formulas in $\mathcal{L}_{\Box\Diamond}$ starting with $\Box$ and $\Diamond$, respectively.

Weak Completeness

Weak completeness will follow from the following lemma which unfortunately has a rather involved proof.

Lemma

$e^F(v, \varphi) = v(\varphi)$ for any $\varphi \in F$ and any $v \in W$.

Theorem

For any finite theory T and formula φ in $\mathcal{L}_{\Box\Diamond}$, $T \models_{GK} \varphi$ implies $T \vdash_{\mathcal{G}_{\Box\Diamond}} \varphi$.

Strong Completeness (I)

To prove strong completeness we utilize compactness of first order classical logic and the following result of Horn:

Lemma

Any countable linear order $(P,<)$ *may be embedded in* $(\mathbb{Q}\cap[0,1],<)$ *preserving all joins and meets existing in* P.

Theorem

(Strong completeness) *For any countable theory* T *and formula* φ *in* $\mathcal{L}_{\Box\Diamond}$, $T\vdash_{\mathcal{G}_{\Box\Diamond}}\varphi$ *if and only if* $T\models_{GK}\varphi$.

Strong Completeness (II)

Sketch of proof: Assume T is countable and $T \nvdash_{\mathcal{G}_{\Box\Diamond}} \varphi$. We define a first order theory T^* with two unary relation symbols W, P, binary $<$, constant symbols 0,1, and c, function symbols $x \circ y$, $S(x, y)$, and $f_\theta(x)$ for each $\theta \in \mathcal{L}_{\Box\Diamond}(V)$ where V is the set of propositional variables of T. By weak completeness will be proved all finite set of T^* is satisfiable. Then, by compactness of first order logic and the downward Löwenheim theorem T^* has a countable model $M^* = (B, W, P, <, 0, 1, a, \circ, S, f_\theta)_{\theta \in \mathcal{L}_{\Box\Diamond}}$. Using Horn's lemma, $(P, <)$ may be embedded in $(\mathbb{Q} \cap [0, 1], <)$ preserving $0, 1$, and all suprema and infima existing in P; therefore, we may assume without loss of generality that the ranges of the functions S and f_θ are contained in $[0, 1]$. Then, it is straightforward to verify that $M = (W, S, e)$, where $e(x, \theta) = f_\theta(x)$ for all $x \in W$, is a wanted GK-model.

Optimal Models

Given a GK-model $M = (W, S, e)$, define a new accessibility relation $S^+xy = S_\Box xy \cdot S_\Diamond xy$, where
$S_\Box xy = \inf_{\varphi \in \mathcal{L}_{\Box\Diamond}} \{e(x, \Box\varphi) \Rightarrow e(y, \varphi)\}$, and
$S_\Diamond xy = \inf_{\varphi \in \mathcal{L}_{\Box\Diamond}} \{e(y, \varphi) \Rightarrow e(x, \Diamond\varphi)\}$, and call M *optimal* if $S^+ = S$.
The following lemma shows that any model is equivalent to an optimal one.

Lemma

(W, S^+, e) is optimal and if e^+ is the extension of e in this model then $e^+(x, \varphi) = e(x, \varphi)$ for any $\varphi \in \mathcal{L}_{\Box\Diamond}$.

Companion Results

Call a GK-frame $\mathcal{M} = \langle W, S \rangle$ *reflexive* if $Sxx = 1$ for all $x \in W$, *transitive* if $Sxy \cdot Syz \leq Sxz$ for all x, y, z, and *symmetric* if $Sxy = Syx$ for all $x, y \in W$.

We can consider the following pairs of modal axioms:

$T_\Box$.	$\Box\varphi \to \varphi$	$T_\Diamond$.	$\varphi \to \Diamond\varphi$	reflexivity
$4_\Box$.	$\Box\varphi \to \Box\Box\varphi$	$4_\Diamond$.	$\Diamond\Diamond\varphi \to \Diamond\varphi$	transitivity
M_1.	$\varphi \to \Box\Diamond\varphi$	M_2.	$\Diamond\Box\varphi \to \varphi$	symmetry

Theorem

Let M be an optimal GK-model, then i) It is reflexive if and only if it validates the schemes $T_\Box + T_\Diamond$. ii) It is transitive if and only if it validates $4_\Box + 4_\Diamond$. iii) It is symmetric if and only if it validates $M_1 + M_2$.

Bi-modal algebras

Definition

An algebras $\mathcal{A} = (G, \wedge, \vee, \rightarrow, 0, 1, I, K)$, shortened as $(\mathcal{G}, I, K)$, is a a bi-modal Gödel algebra, if $\mathcal{G} = (G, \wedge, \vee, \rightarrow, 0, 1)$ is a Gödel algebra and I and K are unary operators on G satisfying the following conditions for all $a, b \in G$:

1 :	$I(a \wedge b) = Ia \wedge Ib$	$K(a \vee b) = Ka \vee Kb$
2 :	$I1 = 1$	$K0 = 0$
3 :	$Ka \rightarrow Ib \leq I(a \rightarrow b)$	$K(a \rightarrow b) \leq Ia \rightarrow Kb$

Then $\mathcal{G}_{\Box\Diamond}$ is the logic given by the variety of *bi-modal Gödel algebras*. This means that $\mathcal{G}_{\Box\Diamond}$ is complete with respect to valuations $v : Var \rightarrow A$ in these algebras, when they are extend to $\mathcal{L}_{\Box\Diamond}$ interpreting $\Box$ and $\Diamond$ by I and K, respectively.

Subvarieties of Bi-modal algebras

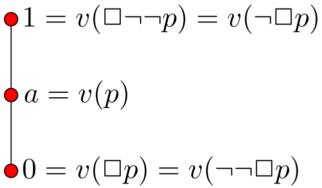
$\mathcal{G}T_{\Box\Diamond}$, $\mathcal{G}S4_{\Box\Diamond}$, and $\mathcal{G}S5_{\Box\Diamond}$ have for algebraic semantic the subvarieties of bi-modal Gödel algebras determined by the corresponding pairs of identities in the following table:

$Ia \leq a$	$a \leq Ka$	reflexivity
$Ia = IIa$	$Ka = KKa$	transitivity
$a \leq IKa$	$KIa \leq a$	symmetry

Notice that the algebraic models of $\mathcal{G}S4_{\Box\Diamond}$ are just the bi-topological pseudo-Boolean algebras of Ono with linear underlying Heyting algebra, and the algebraic models of $\mathcal{G}S5_{\Box\Diamond}$ are the the monadic Heyting algebras of Monteiro and Varsavsky, utilized later by Bull and Fischer Servi to interpret MIPC, with a Gödel basis. It is proper to call them *monadic Gödel algebras*.

Finite Model Property

There is no finite counter-model for the formula $\Box\neg\neg p \rightarrow \neg\neg\Box p$ in Gödel-Kripke semantics. However, the algebra $A = (\{0, a, 1\}, I, K)$ where $\{0 < a < 1\}$ is the three elements Gödel algebra and $I1 = 1$, $Ia = I0 = 0$, $K1 = Ka = 1$, $K0 = 0$ is a bi-modal Gödel algebra (actually a monadic Heyting algebra) providing a finite counterexample to the validity of the formula by means of the valuation $v(p) = a$, as the reader may verify.



Complex Algebras (I)

We may associate to each Gödel-Kripke frame $\mathcal{F} = (W, S)$ a bi-modal Gödel algebra $[0,1]^{\mathcal{F}} = ([0,1]^W, I^{\mathcal{F}}, K^{\mathcal{F}})$ where $[0,1]^W$ is the product Gödel algebra, and for each map $f \in [0,1]^W$:

$$
\begin{aligned}
I^{\mathcal{F}}(f)(w) &= \inf_{w' \in W} (Sww' \Rightarrow f(w')) \\
K^{\mathcal{F}}(f)(w) &= \sup_{w' \in W} (Sww' \cdot f(w'))
\end{aligned}
$$

We call an algebra of the form $[0,1]^{\mathcal{F}}$ a *Gödel complex algebra*.

Complex Algebras (II)

Theorem

$[0,1]^{\mathcal{F}}$ is a bi-modal Gödel algebra, and there is a one to one correspondence between Gödel Kripke models over $\mathcal{F}$, and valuations $v : Var \to [0,1]^{\mathcal{F}}$ given by the adjunction:

$$Var \times W \xrightarrow{e} [0,1] \quad \leftrightarrow \quad Var \xrightarrow{v_e} [0,1]^W, \quad v_e(p) = e(-,p)$$

so that for any formula φ, $v_e(\varphi) = e(\_,\varphi)$.
Moreover, the transformation $\mathcal{F} \longmapsto [0,1]^{\mathcal{F}}$ preserves reflexivity, transitivity and symmetry. Thus, it send Gödel-Kripke frames for $\mathcal{G}T_{\Box\Diamond}$, $\mathcal{G}S4_{\Box\Diamond}$, and $\mathcal{G}S5_{\Box\Diamond}$ into algebraic models for the same logics.

From Bi-modal algebras to GK-models

We associate to each countable bi-modal Gödel algebra A a GK-frame $\mathcal{F}_A$ such that A may be embedded in the associated algebra $[0,1]^{\mathcal{F}_A}$, and to each algebraic valuation η in A a GK-model over $\mathcal{F}_A$ validating the same formulas as η.

Call a theory $T \subseteq \mathcal{L}_{\Box\Diamond}$ *normal* if $T \vdash_{\mathcal{G}_{\Box\Diamond}} \theta$ implies $T \vdash_{\mathcal{G}_{\Box\Diamond}} \Box\theta$ and $T \vdash_{\mathcal{G}_{\Box\Diamond}} \theta \to \rho$ implies $T \vdash_{\mathcal{G}_{\Box\Diamond}} \Diamond\theta \to \Diamond\rho$.

If T is normal, then for each finite fragment F we can obtain the submodel $M_F^T = (W^T, S^F, e^F)$ of the canonical model where $W^T = \{v \in W : v(T) = 1\}$. Hence, if Σ is a finite subset of T such that $\Sigma \not\vdash_{\mathcal{G}_{\Box\Diamond}} \varphi$ there is a canonical model M_F^T such that $e^F(v, \Sigma) = 1$ and $e^F(v, \varphi) < 1$ (take $F \supseteq \Sigma \cup \{\varphi\}$).

Main results

Lemma

If T is a countable normal theory there is GK-model M_T such that $T \vdash_{\mathcal{G}_{\Box\Diamond}} \varphi$ if and only if $M_T \models \varphi$.

Theorem

For any countable bi-modal Gödel algebra A there is Gödel frame $\mathcal{F}_A = (W, S)$ such that:
i) A is embeddable in the Gödel complex algebra $[0,1]^{\mathcal{F}_A}$.
ii) For any valuation $v : Var \to A$ there is a $e_v : W \times Var \to [0,1]$ such that $v(\varphi) = 1$ if and only if $(W, S, e_v) \models \varphi$.

Theorem

The complex algebras generate the variety of bi-modal Gödel algebras.

Conclusions and future works

- We have presented a complete axiomatization of Gödel Modal Logic.
- We would like to find a connection between our fuzzy kripke semantic and classical intuitionists modal semantic.

THANKS!!!

Strong Completeness (III)

The first order theory T^* has the following axioms:

$\forall x \neg (Wx \wedge Px)$

$(P, <)$ is a strict linear order with minimum 0 and maximum 1

$\forall x \forall y (W(x) \wedge W(y) \rightarrow P(S(x, y)))$

$\forall x \forall y (P(x) \wedge P(y) \rightarrow (x \leq y \wedge x \circ y = 1) \vee (x > y \wedge x \circ y = y))$

$\forall x (W(x) \rightarrow f_{\perp}(x) = 0)$

for each $\theta, \psi \in \mathcal{L}_{\Box\Diamond}$:

$\forall x (W(x) \rightarrow P(f_\theta(x)))$

$\forall x (W(x) \rightarrow f_{\theta \wedge \psi}(x) = \min\{f_\theta(x), f_\psi(x)\})$

$\forall x (W(x) \rightarrow f_{\theta \rightarrow \psi}(x) = (f_\varphi(x) \circ f_\psi(x))$

$\forall x (W(x) \rightarrow f_{\Box\theta}(x) = \inf_y (S(x, y) \circ f_\theta(y))$

$\forall x (W(x) \rightarrow f_{\Diamond\theta}(x) = \sup_y (\min\{S(x, y), f_\theta(y)\})$

$W(c) \wedge (f_\varphi(c) < 1)$

for each $\theta \in T : f_\theta(c) = 1$