Two-dimensional logics for (comparative) uncertainty

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In memory of Jon Michael Dunn

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Motivation: Belief based on information

- It is natural to view belief as based on evidence/information
- Potential incompleteness, uncertainty, and contradictoriness of information needs to be dealt with adequately
- Separately, these characteristics has been taken into account by various appropriate logical formalisms and (classical) probability theory
- The first two are often accommodated within one formalism (e.g. imprecise probability), the second two less so.
- Conflict or contradictoriness of information is rather to be *resolved* than to be *reasoned with*.

Two-dimensionality of information

Addressing incompleteness and contradictoriness of information in one framework:

- separating positive and negative information, which are not considered complementary and can overlap
- semantically, distinguishing support for from opposition to a statement (or qualifying/quantifying evidence for and evidence against a statement being the case separately)
- explicit in the double-valuation semantics of Belnap-Dunn logic, and the concept of bi-lattices.
- this approach can be extended to encompass uncertainty measures like *probabilities*, and *graded reasoning*.

Belnap-Dunn logic: qualifying evidence

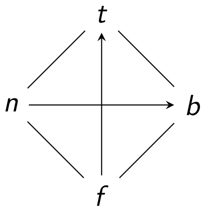
Language L_{BD}:
$\varphi := p \in \mathrm{Prop} \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \neg\varphi$

$(4, \wedge, \vee, \neg)$ is a de Morgan algebra

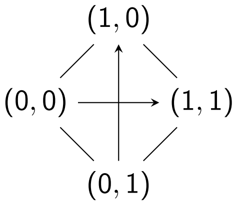
- $(4, \wedge, \vee)$ is a distributive lattice
- each element represents the availability of positive and/or negative information
  - t: true (top)
  - n: no info b: contradictory info
  - f: false (bottom)
- $\neg$ is an involutive de Morgan negation.

BD consequence relation and Exactly true logic

$\Gamma \vDash_{\mathrm{BD}} \varphi$ given as preservation of $\{t, b\}$.
$\Gamma \vDash_{\mathrm{ETL}} \varphi$ given as preservation of $\{t\}$.



Belnap-Dunn square 4



Belnap-Dunn Logic: axiomatics

BD is completely axiomatized using the following axioms and rules:

$\varphi \land \psi \vdash \varphi$ $\quad$ $\varphi \land \psi \vdash \psi$ $\quad$ $\varphi \vdash \psi \lor \varphi$ $\quad$ $\varphi \vdash \varphi \lor \psi$

$\varphi \vdash \neg\neg\varphi$ $\quad$ $\neg\neg\varphi \vdash \varphi$ $\quad$ $\varphi \land (\psi \lor \chi) \vdash (\varphi \land \psi) \lor (\varphi \land \chi)$

$\neg\varphi \land \neg\psi \vdash \neg(\varphi \lor \psi)$ $\quad$ $\neg(\varphi \land \psi) \vdash \neg\varphi \lor \neg\psi$

$$\frac{\varphi \vdash \psi, \ \psi \vdash \chi}{\varphi \vdash \chi} \quad \frac{\varphi \vdash \psi, \ \varphi \vdash \chi}{\varphi \vdash \psi \land \chi} \quad \frac{\varphi \vdash \chi, \ \psi \vdash \chi}{\varphi \lor \psi \vdash \chi} \quad \frac{\varphi \vdash \psi}{\neg\psi \vdash \neg\varphi}$$

- $\Gamma \vdash_{\mathsf{BD}} \varphi$ is the consequence relation generated by the above
- BD is *strongly complete* w.r.t. the algebraic semantics.
- BD allows for a unique (irredundant) DNF and CNF.

Non-standard probabilities: quantifying evidence

= probabilistic extension of BD logic

- $m : P\mathtt{Lit} \to [0,1]$ a mass function: $\sum_{\Gamma \subseteq \mathtt{Lit}} m(\Gamma) = 1$
- Generates an assignment $(p^+, p^-) : L_{\mathrm{BD}} \to [0,1] \times [0,1]^{op}$:

$$p^+(\varphi) = \sum \{m(\Gamma) \mid \Gamma \vdash \varphi\}$$

$$p^-(\varphi) = p^+(\neg\varphi) \quad \text{coherence}$$

The non-standard probability function p^+ satisfies:

(A1)	normalization	$0 \leq p^+(\varphi) \leq 1$
(A2)	monotonicity	if $\varphi \vdash_{\mathrm{BD}} \psi$ then $p^+(\varphi) \leq p^+(\psi)$
(A3)	incl.-excl.	$p^+(\varphi \wedge \psi) + p^+(\varphi \vee \psi) = p^+(\varphi) + p^+(\psi)$.

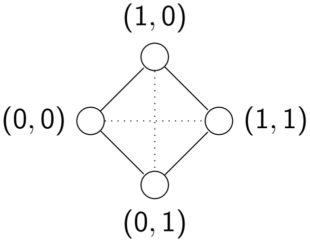
- D. Klein, O. Majer, S. Raffie-Rad, *Probabilities with gaps and gluts*, JPL 2021.
- C. Zhou, Belief functions on distributive lattices. Artif. Intell. 201, (2013).

Non-standard probabilities range

Continuous extension of Belnap-Dunn square: the product bilattice $\mathsf{L}_{[0,1]} \odot \mathsf{L}_{[0,1]}$ with $\mathsf{L}_{[0,1]} = ([0,1], \min, \max)$.

product bilattice $\mathsf{L}_{[0,1]} \odot \mathsf{L}_{[0,1]}$

$$(a_1, a_2) \wedge (b_1, b_2) = (a_1 \wedge b_1, a_2 \vee b_2)$$
$$(a_1, a_2) \vee (b_1, b_2) = (a_1 \vee b_1, a_2 \wedge b_2)$$
$$\neg(a_1, a_2) = (a_2, a_1)$$



- $(p^+(\varphi), p^-(\varphi))$: positive/negative probabilistic support of φ. $(0,0)$: no information available, $(1,1)$: maximally conflicting information
- "classical" vertical line: $p^+(\varphi) = 1 - p^-(\varphi)$
- Graded reasoning about (belief based on) probabilities?

Other uncertainty measures

Aggregating probabilities: (p^+, p^-), in general satisfy only

(A1) normalization	$0 \leq p^+(\varphi) \leq 1$
(A2) monotonicity	if $\varphi \vdash_{\mathsf{BD}} \psi$ then $p^+(\varphi) \leq p^+(\psi)$
coherence	$p^-(\varphi) = p^+(\neg\varphi)$

Inner probabilities (cf. belief functions): (b^+, b^-)

(A1) normalization	$0 \leq b^+(\varphi) \leq 1$
(A2) monotonicity	if $\varphi \vdash_{\mathsf{BD}} \psi$ then $b^+(\varphi) \leq b^+(\psi)$
(A3) incl.-excl.	$b^+(\varphi \vee \psi) \geq b^+(\varphi) + b^+(\psi) - b^+(\varphi \wedge \psi)$
coherence	$b^-(\varphi) = b^+(\neg\varphi)$

Necessity/possibility measures: $\nu = (n, p), \pi = (p, n)$

normalization, monotonicity

distribution $n(\varphi \wedge \psi) = n(\varphi) \wedge n(\psi), p(\varphi \vee \psi) = p(\varphi) \vee p(\psi)$

coherence $\nu(\neg\varphi) = \neg\pi(\varphi)$

Two-dimensional logics for uncertainty

- to be interpreted over an algebra (matrix) expanding $[0,1] \odot [0,1]$ with implication, fusion, negation, ...
- to be able to express:
  I. all three probability (belief functions) axioms in case of uncertainty
  — derived from Łukasiewicz logic and $[0,1]_{Ł}$
  II. monotonicity and coherence (and $\wedge, \vee$ distribution) in case of comparative uncertainty
  — derived from Gödel logic and $[0,1]_G$
- two ways of negating implication
  (a) "de Morgan" way, using a co-implication

$$\neg(a \rightarrow b) := (\neg b \prec \neg a)$$

  (b) "Nelson" way, combining positive and negative semantical values

$$\neg(a \rightharpoonup b) := (a \mathbin{\&} \neg b)$$

case I.(a): $Ł^2(\to)$, reasoning with probabilities

Standard MV algebra

$[0,1]_Ł = ([0,1], \wedge, \vee, \&_Ł, \to_Ł)$:

$$a \wedge b := \min\{a,b\}, \qquad a \&_Ł b := \max\{0, a+b-1\}$$
$$a \vee b := \max\{a,b\} \qquad a \to_Ł b := \min\{1, 1-a+b)\}$$
$$\sim_Ł a := a \to_Ł 0 = 1-a$$

Definable connectives:

$$a \oplus_Ł b := \sim a \to_Ł b = \min\{1, a+b\}$$
$$a \ominus_Ł b := \sim(a \to_Ł b) = \max\{0, a-b\}$$

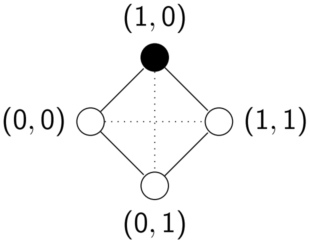
$\ominus_Ł$ can be seen as a co-implication.

case I.(a): $Ł^2_{(1,0)}(\to)$, reasoning with probabilities

Bilattice product $[0,1] \odot [0,1]$**,** $\neg(a_1, a_2) = (a_2, a_1)$**,**
$F = \{(1,0)\}$**,**

$[0,1] \odot [0,1]$ expanded with

$$(a_1, a_2) \to (b_1, b_2) = (a_1 \to_Ł b_1, b_2 \ominus a_2)$$
$$(a_1, a_2)\&(b_1, b_2) = (a_1 \&_Ł b_1, a_2 \oplus b_2)$$
$$\sim(a_1, a_2) = (\sim_Ł a_1, \sim_Ł a_2)$$



Notice: $\neg$ is symmetry along the horizontal, $\sim$ is symmetry along the middle point, $\sim\neg$ is symmetry along the vertical (conflation). $\neg\alpha \leftrightarrow \sim\alpha$ defines the vertical. $\neg$ and $\sim$ are distinct.

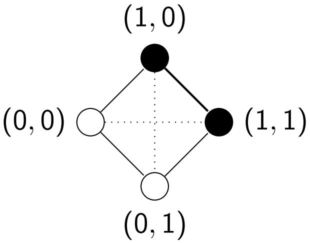
$\Gamma \models_{Ł^2_{(1,0)}(\to)} \alpha$ defined as preservation of $(1,0)$.
Its $(\wedge, \vee, \neg)$-fragment coincides with ETL.

case I.(a): $Ł^2_{(1,1)^\uparrow}(\to)$, reasoning with probabilities

Bilattice product $[0,1] \odot [0,1]$, $\neg(a_1, a_2) = (a_2, a_1)$, $F = (1,1)^\uparrow$,

$[0,1] \odot [0,1]$ expanded with

$$(a_1, a_2) \to (b_1, b_2) = (a_1 \to_Ł b_1, b_2 \ominus a_2)$$
$$(a_1, a_2) \& (b_1, b_2) = (a_1 \&_Ł b_1, a_2 \oplus b_2)$$
$$\sim(a_1, a_2) = (\sim_Ł a_1, \sim_Ł a_2)$$



Notice: $\neg$ is symmetry along the horizontal, $\sim$ is symmetry along the middle point, $\sim\neg$ is symmetry along the vertical (conflation). $\neg\alpha \leftrightarrow \sim\alpha$ defines the vertical. $\neg$ and $\sim$ are distinct.

$\Gamma \models_{Ł^2_{(1,1)^\uparrow}(\to)} \alpha$ defined as preservation of $(1,1)^\uparrow$.
Its $(\wedge, \vee, \neg)$-fragment coincides with BD.

case I.(a): $Ł^2_{(1,0)}(\to)$, reasoning with probabilities

$Ł^2_{(1,0)}(\to)$: Ł **expanded with the bi-lattice negation** $\neg$.

Axiomatization of $Ł^2_{(1,0)}(\to)$

$$\alpha \to (\beta \to \alpha)$$
$$(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))$$
$$((\alpha \to \beta) \to \beta) \to ((\beta \to \alpha) \to \alpha)$$
$$(\sim\beta \to \sim\alpha) \to (\alpha \to \beta)$$

$$\neg\neg\alpha \leftrightarrow \alpha$$
$$\neg{\sim}\alpha \leftrightarrow {\sim}\neg\alpha$$
$$({\sim}\neg\alpha \to {\sim}\neg\beta) \leftrightarrow {\sim}\neg(\alpha \to \beta)$$
$$\alpha, \alpha \to \beta \vdash \beta \quad \alpha \vdash {\sim}\neg\alpha$$

- $\neg$- negation normal form
- Deduction theorem:
  $\Gamma, \alpha \vdash_{Ł^2_{(1,0)}(\to)} \beta$ iff $\exists n$, $n\ \Gamma \vdash_{Ł^2_{(1,0)}(\to)} ({\sim}\neg\alpha)^n \,\&\, \alpha^m \to \beta$

Theorem: $Ł^2_{(1,0)}(\to)$ is **finitely strongly standard–complete** w.r.t. $(([0,1] \odot [0,1], \to, \sim), \{(1,0)\})$

Axiomatization of $Ł^2_{(1,1)^\uparrow}(\to)$

$$\alpha \to (\beta \to \alpha)$$
$$(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))$$
$$((\alpha \to \beta) \to \beta) \to ((\beta \to \alpha) \to \alpha)$$
$$(\sim\beta \to \sim\alpha) \to (\alpha \to \beta)$$

$$\neg\neg\alpha \leftrightarrow \alpha$$
$$\neg\sim\alpha \leftrightarrow \sim\neg\alpha$$
$$(\sim\neg\alpha \to \sim\neg\beta) \leftrightarrow \sim\neg(\alpha \to \beta)$$
$$\alpha, \alpha \to \beta \vdash \beta \quad \vdash \alpha / \vdash \sim\neg\alpha$$

- $\neg$- negation normal form
- Deduction theorem:
  $\Gamma, \alpha \vdash_{Ł^2_{(1,1)^\uparrow}(\to)} \beta$ iff $\exists n$ $\Gamma \vdash_{Ł^2_{(1,1)^\uparrow}(\to)} \alpha^n \to \beta$

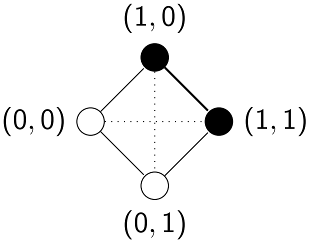
Theorem: $Ł^2_{(1,1)^\uparrow}(\to)$ is **finitely strongly standard–complete** w.r.t. $(([0,1] \odot [0,1], \to, \sim), (1,1)^\uparrow)$

Case I (b): $Ł^2_{(1,1)^\uparrow}(\rightarrowtriangle)$, reasoning with probabilities

Product bi-lattice $[0,1] \odot [0,1]$, $\neg(a_1, a_2) = (a_2, a_1)$, $F = (1,1)^\uparrow$:

$[0,1] \odot [0,1]$ expanded with

$$(a_1, a_2) \rightarrowtriangle (b_1, b_2) = (a_1 \rightarrow_Ł b_1, a_1 \&_Ł b_2)$$
$$(a_1, a_2) \& (b_1, b_2) = (a_1 \&_Ł b_1, a_1 \rightarrow_Ł \sim_Ł b_1)$$
$$\sim(a_1, a_2) = (\sim_Ł a_1, a_1)$$



$\Gamma \models_{Ł^2_{(1,1)^\uparrow}(\rightarrowtriangle)} \alpha$ defined as preservation of $F = \{(1, a) \mid a \in [0,1]\}$.
Its $(\wedge, \vee, \neg)$-fragment coincides with BD.

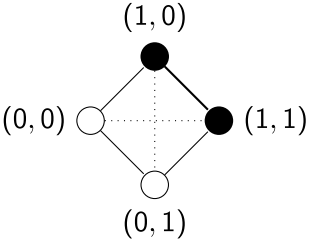
The weak equivalence $\alpha \leftrightarrowtriangle \beta := (\alpha \rightarrowtriangle \beta) \wedge (\beta \rightarrowtriangle \alpha)$ is not congruential, the strong one $\alpha \Longleftrightarrow \beta := (\alpha \leftrightarrowtriangle \beta) \wedge (\neg\alpha \leftrightarrowtriangle \neg\beta)$ is.

Case I (b): $Ł^2_{(1,1)^\uparrow}(\rightarrowtriangle)$, reasoning with probabilities

Product bi-lattice $[0,1] \odot [0,1]$, $\neg(a_1, a_2) = (a_2, a_1)$, $F = (1,1)^\uparrow$:

$[0,1] \odot [0,1]$ expanded with

$$(a_1, a_2) \rightarrowtriangle (b_1, b_2) = (a_1 \to_Ł b_1, a_1 \&_Ł b_2)$$
$$(a_1, a_2) \& (b_1, b_2) = (a_1 \&_Ł b_1, a_1 \to_Ł \sim_Ł b_1)$$
$$\sim(a_1, a_2) = (\sim_Ł a_1, a_1)$$



$\Gamma \models_{Ł^2_{(1,1)^\uparrow}(\rightarrowtriangle)} \alpha$ defined as preservation of $F = \{(1,a) \mid a \in [0,1]\}$.
Its $(\wedge, \vee, \neg)$-fragment coincides with BD.

$\sim\alpha$ is always on the vertical. $\sim\alpha \leftrightarrowtriangle \neg\alpha$ defines the vertical, $\sim\alpha \rightarrowtriangle \neg\alpha$ defines the right triangle, and $\neg\alpha \rightarrowtriangle \sim\alpha$ the left. $(\alpha \rightarrowtriangle \beta) \wedge (\neg\alpha \rightarrowtriangle \neg\beta)$ captures the information order.

Case I (b): $Ł^2_{(1,1)^\uparrow}(\rightarrowtriangle)$, reasoning with probabilities

Axiomatics of $Ł^2_{(1,1)^\uparrow}(\rightarrowtriangle)$:

The axioms of Łukasiewicz logic (in terms of the weak implication) with MP as the only rule, plus the $\neg$-axioms:

$$\neg\neg\alpha \leftrightarrowtriangle \alpha$$
$$\neg(\alpha \wedge \beta) \leftrightarrowtriangle \neg\alpha \vee \neg\beta$$
$$\neg(\alpha \vee \beta) \leftrightarrowtriangle \neg\alpha \wedge \neg\beta$$
$$\neg(\alpha \rightarrow \beta) \leftrightarrowtriangle (\alpha \& \neg\beta)$$
$$\neg(\alpha \& \beta) \leftrightarrowtriangle (\alpha \rightarrow \sim\beta)$$
$$\neg 0 \leftrightarrowtriangle \sim 0$$

- $\neg$-negation normal form (weakly equivalent only)
- Deduction theorem as in Ł
- Finite strong standard completeness (FSSC)

case II.(a): $G^2_{(1,0)}(\to)$, comparative uncertainty

Standard Gödel algebra:

$[0,1]_G = ([0,1], \wedge, \vee, \to_G)$

$$a \to_G b = \begin{cases} 1, & \text{if } a \leq b \\ b & \text{else} \end{cases} \qquad \sim_G a := a \to_G 0$$

$$c \leq a \to_G b \quad \text{iff} \quad a \wedge c \leq b$$

can be expanded by a co-implication:

$$b \prec_G a = \begin{cases} 0, & \text{if } b \leq a \\ b & \text{else} \end{cases} \qquad -_G a := 1 \prec_G a$$

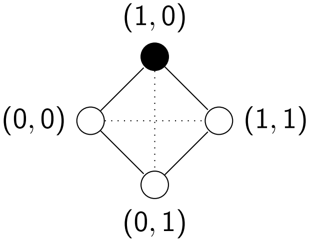
$$b \prec_G a \leq c \quad \text{iff} \quad b \leq a \vee c$$

case II.(a): $G^2_{(1,0)}(\to)$, comparative uncertainty

Product bilattice $[0,1] \odot [0,1]$, $\neg(a_1,a_2) = (a_2,a_1)$,
$F = \{(1,0)\}$,

$[0,1] \odot [0,1]$ expanded with

$$(a_1,a_2) \to (b_1,b_2) = (a_1 \to_G b_1, b_2 \prec_G a_2)$$
$$\sim(a_1,a_2) = (\sim_G a_1, -a_2)$$



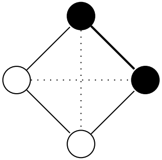
$\Gamma \vDash_{G^2_{(1,0)}(\to)} \alpha$ defined as preservation of $(1,0)$.
Its $(\wedge, \vee, \neg)$-fragment coincides with ETL.

case II.(a): $G^2_{(1,1)^\uparrow}(\to)$, comparative uncertainty

Product bilattice $[0,1] \odot [0,1]$, $\neg(a_1, a_2) = (a_2, a_1)$, $F = (1,1)^\uparrow$,

$[0,1] \odot [0,1]$ expanded with

$$(a_1, a_2) \to (b_1, b_2) = (a_1 \to_G b_1, b_2 \prec_G a_2)$$
$$\sim(a_1, a_2) = (\sim_G a_1, -a_2)$$



$\Gamma \vDash_{G^2_{(1,1)^\uparrow}(\to)} \alpha$ defined as preservation of $(1,1)^\uparrow$.
Its $(\wedge, \vee, \neg)$-fragment coincides with BD.

case II.(a): $G^2_{(1,0)}(\to)$, comparative uncertainty

$G^2_{(1,0)}(\to)$: **bi-Gödel logic expanded with a bi-lattice negation**

Axiomatization: bi-IL in the language $\{\wedge, \vee, \to, \prec, 0, 1\}$ extended with the prelinearity axiom: $(\alpha \to \beta) \vee (\beta \to \alpha)$

$$\neg\neg\alpha \leftrightarrow \alpha \quad \neg 0 \leftrightarrow {\sim} 0$$
$$\neg(\alpha \wedge \beta) \leftrightarrow (\neg\alpha \vee \neg\beta)$$
$$\neg(\alpha \vee \beta) \leftrightarrow (\neg\alpha \wedge \neg\beta)$$
$$\neg(\alpha \to \beta) \leftrightarrow (\neg\beta \prec \neg\alpha)$$
$$\alpha \vdash {\sim}\neg\alpha$$

- $\neg$-negation normal form; $p \wedge \neg p \vdash q$
- Deduction theorem: $\Gamma, \alpha \vdash \beta$ iff $\Gamma \vdash {\sim}{-}\alpha \wedge {\sim}\neg\alpha \to \beta$
- Standard strong completeness (SSC)
- **Its theorems coincide with Wansing's I_4C_4 extended with prelinearity axiom.**

case II.(a): $G^2_{(1,1)^\uparrow}(\rightarrow)$, comparative uncertainty

Axiomatization: bi-IL in the language $\{\wedge, \vee, \rightarrow, \prec, 0, 1\}$ extended with the prelinearity axiom: $(\alpha \rightarrow \beta) \vee (\beta \rightarrow \alpha)$

$$\neg\neg\alpha \leftrightarrow \alpha \quad \neg 0 \leftrightarrow {\sim} 0$$

$$\neg(\alpha \wedge \beta) \leftrightarrow (\neg\alpha \vee \neg\beta)$$

$$\neg(\alpha \vee \beta) \leftrightarrow (\neg\alpha \wedge \neg\beta)$$

$$\neg(\alpha \rightarrow \beta) \leftrightarrow (\neg\beta \prec \neg\alpha)$$

$$\vdash \alpha \;/\; \vdash {\sim}\neg\alpha$$

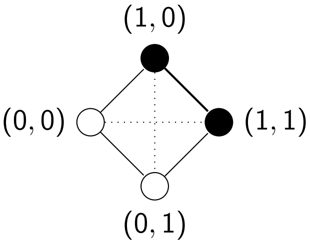
- $\neg$-negation normal form; $p \wedge \neg p \nvdash q$
- Deduction theorem: $\Gamma, \alpha \vdash \beta$ iff $\Gamma \vdash {\sim}{-}\alpha \rightarrow \beta$
- Standard strong completeness (SSC)
- **= Wansing's I_4C_4 extended with prelinearity axiom.**

Case II (b): $G^2_{(1,1)^\uparrow}(\rightarrowtriangle)$, comparative uncertainty

product bi-lattice $[0,1] \odot [0,1]$, $\neg(a_1, a_2) = (a_2, a_1)$, $F = (1,1)^\uparrow$:

$[0,1] \odot [0,1]$ expanded with

$$(a_1, a_2) \rightarrowtriangle (b_1, b_2) = (a_1 \to_G b_1, a_1 \wedge b_2)$$
$$\sim(a_1, a_2) = (\sim_G a_1, a_1)$$



$\Gamma \models_{G^2_{(1,1)^\uparrow}(\rightarrowtriangle)} \alpha$ defined as preservation of $F = \{(1,a) \mid a \in [0,1]\}$.

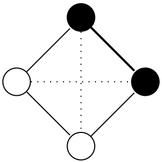
The weak equivalence $\alpha \leftrightarrowtriangle \beta := (\alpha \rightarrowtriangle \beta) \wedge (\beta \rightarrowtriangle \alpha)$ is not congruential, the strong one $\alpha \Longleftrightarrow \beta := (\alpha \leftrightarrowtriangle \beta) \wedge (\neg\alpha \leftrightarrowtriangle \neg\beta)$ is.

Case II (b): $G^2_{(1,1)^\uparrow}(\rightarrowtriangle)$, comparative uncertainty

product bi-lattice $[0,1] \odot [0,1]$**,** $\neg(a_1, a_2) = (a_2, a_1)$**,** $F = (1,1)^\uparrow$:

$[0,1] \odot [0,1]$ expanded with

$$(a_1, a_2) \rightarrowtriangle (b_1, b_2) = (a_1 \rightarrow_G b_1, a_1 \wedge b_2)$$
$$\sim(a_1, a_2) = (\sim_G a_1, a_1)$$



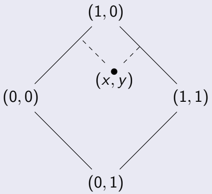
$\Gamma \models_{G^2_{(1,1)^\uparrow}(\rightarrowtriangle)} \alpha$ defined as preservation of $F = \{(1,a) \mid a \in [0,1]\}$.

The resulting logic coincides with Nelson's $N4^\perp$ extended with prelinearity (global consequence).

Two-dimensional logics: summing up

… of quantified uncertainty

- $Ł^2_{(1,1)^{\uparrow}}(\rightarrowtriangle)$, $Ł^2_{(1,0)^{\uparrow}}(\rightarrow)$, $Ł^2_{(1,1)^{\uparrow}}(\rightarrow)$
- Finitely strong standard complete axiomatization
- Varying the filters $(x, y)^{\uparrow}$: different tautologies, different entailments
- Constraint tableaux calculi, finitary entailment is coNP-complete.

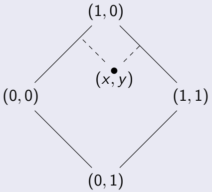


- M. Bílková, S. Fritella, D. Kozhemiachenko. *Constraint tableaux for two-dimensional fuzzy logics*, TABLEAUX 2021.

Two-dimensional logics: summing up

… of comparative uncertainty

- $G^2_{(1,0)^\uparrow}(\rightarrow)$, $G^2_{(1,1)^\uparrow}(\rightarrow)$, $G^2_{(1,1)^\uparrow}(\rightarrowtriangle)$
- Strong standard complete axiomatization
- Varying the filters $(x,y)^\uparrow$: same tautologies, different entailments:
- Constraint tableaux calculi, frame semantics, finitary entailment is coNP-complete.



$1 > x > y > 0$ for $G^2(\rightarrow)$

$\models_{(x,1)^\uparrow} \subset \models_{(1,1)^\uparrow} \subset \models_{(1,0)^\uparrow}$

$\models_{(x,1)^\uparrow} \subset \models_{(y,x)^\uparrow} \subset \models_{(x,x)^\uparrow} \subset \models_{(x,y)^\uparrow} \subset \models_{(1,0)^\uparrow}$

An application: two-layer logics of belief

Probabilistic belief (quantified uncertainty)

- Two-layer logics $(\mathsf{BD}, M, Ł^2_{(1,0)}(\to))$ or $(\mathsf{BD}, M, Ł^2_{(1,1)^{\uparrow}}(\rightarrowtriangle))$
- Finite strong completeness w.r.t. intended semantics

M: $\vdash_{Ł^2_{(1,0)}(\to)} B\neg\varphi \leftrightarrow \neg B\varphi \qquad \varphi \vdash_{\mathsf{BD}} \psi \;/\; \vdash_{Ł^2_{(1,0)}(\to)} B\varphi \to B\psi$

$\vdash_{Ł^2_{(1,0)}(\to)} B(\varphi \vee \psi) \leftrightarrow (B\varphi \ominus B(\varphi \wedge \psi)) \oplus B\psi$

- M.Bílková, S. Fritella, O. Majer, S. Nazari. *Belief based on inconsistent information*, DaLi 2020, LNCS volume 12569, pp 68-86, 2020.

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An application: two-layer logics of belief

Monotone coherent belief (comparative uncertainty)

- Two-layer logics (BD, M', $G^2_{(1,0)}(\to)$) or (BD, M', $G^2_{(1,1)^\uparrow}(\rightarrowtriangle)$)
- Strong completeness w.r.t. intended semantics

Modal rules M': $\vdash_{G^2_{(1,0)}(\to)} B\neg\varphi \leftrightarrow \neg B\varphi$

$$\varphi \vdash_{\mathsf{BD}} \psi \;/\; \vdash_{G^2_{(1,0)}(\to)} B\varphi \to B\psi.$$

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Monotone coherent belief (comparative uncertainty)

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Modal rules M': $\vdash_{G^2} B\neg\varphi \Leftrightarrow \neg B\varphi$

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Example: believing $(\varphi \wedge \neg\varphi)$

For a BD formula φ,

$Ł^2_{(1,1)^\uparrow}(\rightarrowtriangle)$

- $B(\varphi \wedge \neg\varphi) \rightarrowtriangle \sim(B(\varphi \wedge \neg\varphi))$ says "rather small degree of conflict" (closer to 0 then 1)
- $\sim B(\varphi \wedge \neg\varphi) \rightarrowtriangle (B(\varphi \wedge \neg\varphi))$ says "rather big degree of conflict" (closer to 1 then 0)

$Ł^2_{(1,0)}(\rightarrow)$

- $B(\varphi \wedge \neg\varphi) \rightarrow \sim(B(\varphi \wedge \neg\varphi))$ says "rather small degree of conflict" and "rather small degree of ignorance"

(By "says" I mean consequences of the formula being designated in the resp. algebra.)

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