Connecting dualities for compact Hausdorff spaces

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- Gelfand duality (1940s)
- De Vries duality (1962)
- Isbell duality (1972)

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We want to characterize the frames of the form $\mathrm{Op}(X)$ for some $X \in \mathbf{KHaus}$.

Op(X) is compact and regular

Since X is compact

if $\bigcup_{i \in I} U_i = X$, then there exist $i_1, \ldots, i_n \in I$ such that $U_{i_1} \cup \cdots \cup U_{i_n} = X$.

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$$U \prec V \quad \text{iff} \quad \mathrm{cl}(U) \subseteq V$$

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If we denote by $U^* = \text{int}(X \setminus U)$ the pseudocomplement of U in Op(X), then $U \prec V$ iff $U^* \cup V = X$.

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If $X \in$ **KHaus**, then it is regular. So each open subset V can be written as the union of all the opens that are well-inside V, i.e.

$$V = \bigcup \{ U \in \mathrm{Op}(X) \mid U \prec V \}$$

KRFrm

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Let L be a frame.

- We say L is compact if for any $S \subseteq L$, that $\bigvee S = 1_L$ implies that there is a finite subset $S' \subseteq S$ such that $\bigvee S' = 1_L$.

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If $f : X \to Y$ is a continuous map, then $f^{-1} : \mathrm{Op}(Y) \to \mathrm{Op}(X)$ is a frame homomorphism. Thus

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Proposition

$\mathrm{Op} : \mathbf{KHaus} \to \mathbf{KRFrm}$ *is a contravariant functor.*

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Points of L can be equivalently defined as frame homomorphisms $L \to 2$ or as meet-prime elements of L.

pt(L)

Definition

Let L be a frame. We can define a topology on pt(L) whose opens are

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Proposition

If $L \in \mathbf{KRFrm}$, *then* $\mathrm{pt}(L) \in \mathbf{KHaus}$.

If $\alpha : L \to M$ is a frame homomorphism, the inverse image function $\alpha^{-1} : \mathrm{pt}(M) \to \mathrm{pt}(L)$ is a continuous function.

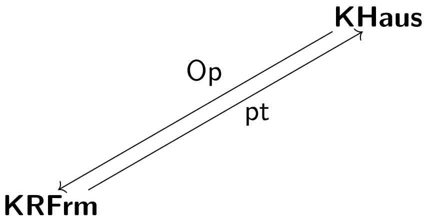
Proposition

$\mathrm{pt} : \mathbf{KRFrm} \to \mathbf{KHaus}$ *is a contravariant functor.*

Isbell duality

Theorem (Isbell 1972)

The contravariant functors Op *and* pt *give rise to a dual equivalence between* **KHaus** *and* **KRFrm**.


KHaus
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Proposition (Tarski)

$\mathrm{RO}(X)$ *ordered by inclusion is a complete boolean algebra where*

$$\bigvee \mathcal{S} = \mathrm{int}\left(\mathrm{cl}\left(\bigcup \mathcal{S}\right)\right)$$

$$\neg U = \mathrm{int}(X \setminus U)$$

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We also need to consider the well-inside relation restricted to $\mathrm{RO}(X)$. That is, $U \prec V$ iff $\mathrm{cl}(U) \subseteq V$.

Well-inside relation on RO(X)

Let $X \in \mathbf{KHaus}$. The complete boolean algebra structure on RO(X) is not enough to recover the space X.

We also need to consider the well-inside relation restricted to RO(X). That is, $U \prec V$ iff $\mathrm{cl}(U) \subseteq V$.

Proposition

- $\emptyset \prec \emptyset$,
- $U \prec V \;\Rightarrow\; U \subseteq V$,
- $U \subseteq V \prec W \subseteq O \;\Rightarrow\; U \prec O$,
- $U \prec V,\, U \prec W \;\Rightarrow\; U \prec V \cap W$,
- $U \prec V \;\Rightarrow\; \neg V \prec \neg U$
- $U \prec V \;\Rightarrow\; \exists W \in \mathrm{RO}(X)$ *such that* $U \prec W \prec V$,
- $V \neq \emptyset \;\Rightarrow\; \exists W \in \mathrm{RO}(X) \setminus \{\emptyset\}$ *such that* $W \prec V$.

De Vries algebras

Definition

A de Vries algebra is a complete boolean algebra B together with a relation $\prec$ such that

- $0 \prec 0$,
- $a \prec b \;\Rightarrow\; a \leq b$,
- $a \leq b \prec c \leq d \;\Rightarrow\; a \prec d$,
- $a \prec b$, $a \prec c \;\Rightarrow\; a \prec b \wedge c$,
- $a \prec b \;\Rightarrow\; \neg b \prec \neg a$,
- $a \prec b \;\Rightarrow\; \exists c \in B$ such that $a \prec c \prec b$,
- $b \neq 0 \;\Rightarrow\; \exists c \in B \setminus \{0\}$ such that $c \prec b$.

DeV

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Proposition

RO : **KHaus** $\to$ **DeV** *is a contravariant functor.*

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Proposition

For each $x \in X$, the set E_x is maximal among proper round filters of $\mathrm{RO}(X)$.

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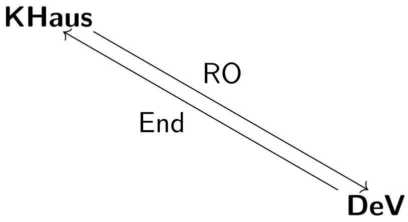
Theorem (De Vries 1962)

- *If B is a de Vries algebra, then* $\mathrm{End}(B) \in \mathbf{KHaus}$.
- *The contravariant functors* RO *and* End *give rise to a dual equivalence between* **KHaus** *and* **DeV**.



KHaus
RO
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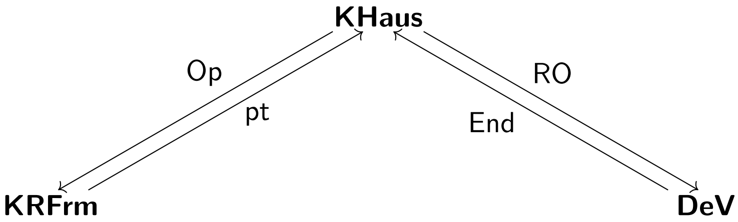
KHaus
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Similar approaches were developed by the Krein brothers, Kakutani, Yosida, Henriksen and Johnson.

Algebra of continuous functions $C(X)$

Definition

Let X be a compact Hausdorff space.
We denote by $C(X)$ the set of continuous real-valued functions on X.

With the partial order $f \leq g$ iff $f(x) \leq g(x)$ for each $x \in X$ and pointwise addition, multiplication, and scalar multiplication:

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- for each $f \in C(X)$, if $f \leq 1/n$ for each $n \in \mathbb{N}$, then $f \leq 0$ (we say $C(X)$ is archimedean).

Bounded archimedean ℓ-algebras

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Let $\mathbf{ba\ell}$ be the category of bounded archimedean ℓ-algebras and unital ℓ-algebra homomorphisms.

Let $h : X \to Y$ be a continuous function between compact Hausdorff spaces. The map $C(h) : C(Y) \to C(X)$ associating to $f \in C(Y)$ the function $f \circ h \in C(X)$ is a $\mathbf{ba\ell}$-morphism.

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We want to characterize the $A \in \mathbf{ba\ell}$ of the form $C(X)$ for some $X \in \mathbf{KHaus}$.

Uniformly complete bounded archimedean ℓ-algebras

Definition

- We can define a norm on each $A \in \mathbf{ba}\boldsymbol{\ell}$ by

$$\|a\| = \inf\{\lambda \in \mathbb{R} \mid |a| \leq \lambda \cdot 1\}.$$

Uniformly complete bounded archimedean ℓ-algebras

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- We can define a norm on each $A \in \mathbf{ba}\boldsymbol{\ell}$ by

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Proposition

$C(X) \in \mathbf{uba\ell}$ for each $X \in \mathbf{KHaus}$.

Points as maximal ℓ-ideals

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For each $x \in X$ the set M_x is maximal among the proper ℓ-ideals of $C(X)$.

The Yosida space

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- Let $\alpha : A \to B$ be a unital ℓ-algebra homomorphism.
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  This defines a contravariant functor $Y : \mathbf{ba\ell} \to \mathbf{KHaus}$.

Adjunction and duality

Theorem

There is a dual adjunction between $\mathbf{ba\ell}$ *and* **KHaus**

$$\mathbf{ba\ell} \underset{Y}{\overset{C}{\leftrightarrows}} \mathbf{KHaus}$$

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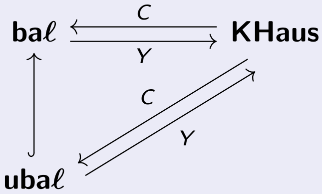
There is a dual adjunction between **baℓ** *and* **KHaus** *This adjunction restricts to a dual equivalence between* **ubaℓ** *and* **KHaus***.*



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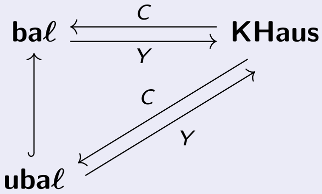

ubaℓ *is a reflective subcategory of* **baℓ** *and* CY : **baℓ** $\to$ **ubaℓ** *is a reflector.*

Key ingredients

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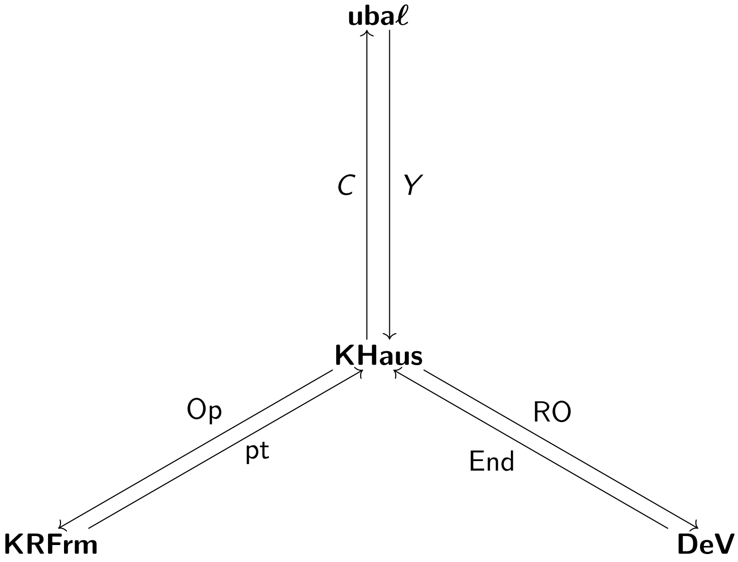
Stone-Weierstrass theorem is used to show that each $A \in \mathbf{ba}\ell$ embeds into $C(Y_A)$ as a uniformly dense subalgebra.

$$\begin{array}{c}\mathbf{uba}\ell \\ C \Big\uparrow \Big\downarrow Y \\ \mathbf{KHaus}\end{array}$$



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Connecting the dualities

It is a consequence of Isbell, de Vries, and Gelfand dualities that $\mathbf{uba}\boldsymbol{\ell}$, **KRFrm**, and **DeV** are equivalent categories.

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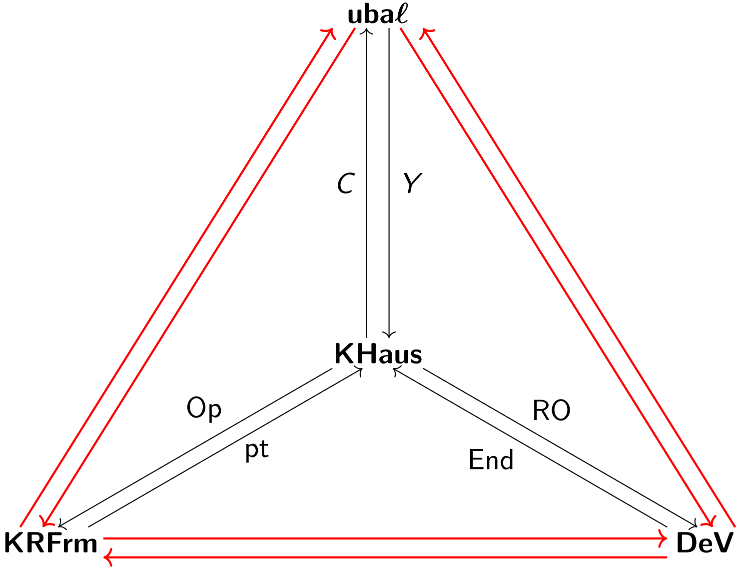
Our goal is to connect these dualities by establishing equivalences between **uba**$\boldsymbol{\ell}$, **KRFrm**, and **DeV** using point-free and choice-free methods.



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From **KRFrm** to **DeV**

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If U^* is the pseudocomplement of U in $\mathrm{Op}(X)$, then

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If $L \in \mathbf{KRFrm}$, an element $a \in L$ is regular if $a^{**} = a$. The set $\mathfrak{B}(L)$ of regular elements of L is called the booleanization of L.

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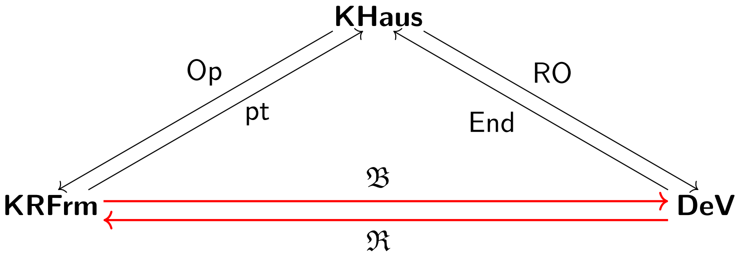
Theorem

The functors $\mathfrak{B}$: **KRFrm** $\to$ **DeV** *and* $\mathfrak{R}$: **DeV** $\to$ **KRFrm** *give rise to an equivalence between* **KRFrm** *and* **DeV**.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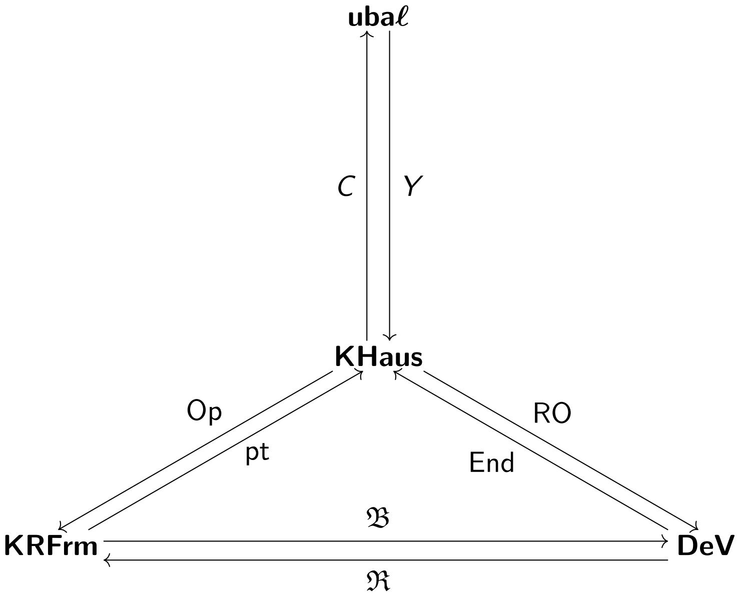

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We want to describe the opens of X in terms of $C(X)$. If U is open of X, then the set of continuous function vanishing on $X \setminus U$ form an archimedean ℓ-ideal of $C(X)$.

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Theorem

- *The set* $\mathrm{Arch}(A)$ *of all archimedean* ℓ*-ideals of* A *ordered by inclusion forms a compact regular frame.*
- *This yields a covariant functor* $\mathrm{Arch} : \mathbf{uba\ell} \to \mathbf{KRFrm}$.

From $\mathbf{ubal}$ to $\mathbf{DeV}$

To characterize the regular opens of X in terms of $C(X)$ it is sufficient to describe the regular elements of $\mathrm{Arch}(A)$. It turns out that they are exactly the annihilator ideals.

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An ℓ-ideal I of $A \in \mathbf{ba}\ell$ is called an annihilator ideal if it is of the form $\mathrm{ann}(S) = \{a \in A \mid as = 0 \text{ for all } s \in S\}$ for some $S \subseteq A$.

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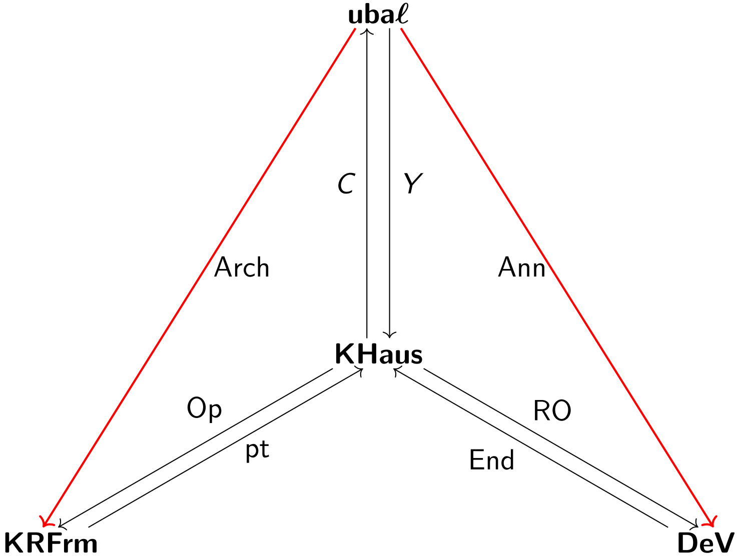
Theorem

- *The set of all annihilator ideals of A ordered by inclusion together with the relation $I \prec J$ iff $\mathrm{ann}(I) + J = A$ forms a de Vries algebra.*
- *This yields a covariant functor $\mathrm{Ann} : \mathbf{uba}\ell \to \mathbf{DeV}$.*



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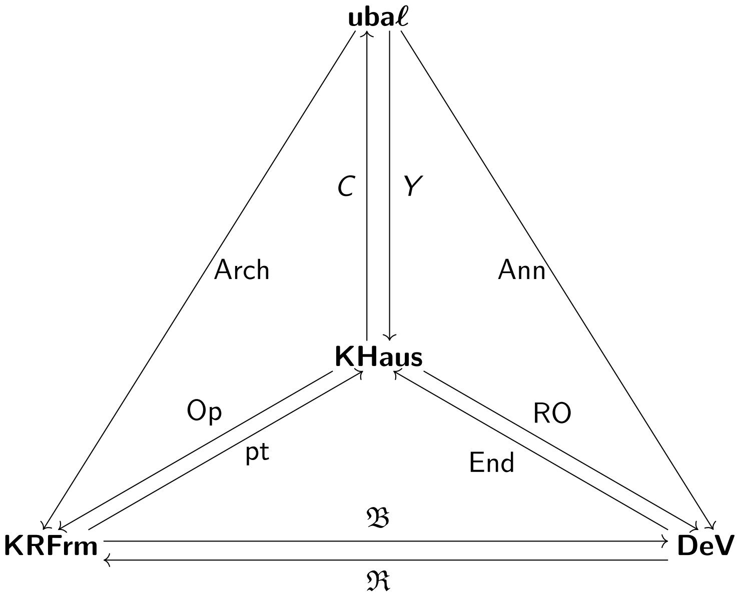

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From **KRFrm** to **ubaℓ**

Two ways to go in this direction.

From **KRFrm** to $\mathbf{uba}\ell$

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We showed that $D(\mathbb{R}[B(L)])$ is the canonical extension of A.

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- We obtain $A \in$ **ubaℓ** as the set of reflexive elements of $D(\mathbb{R}[B])$ with respect to the proximity relation obtained by lifting $\prec$ of B.

From **DeV** to **ubaℓ**

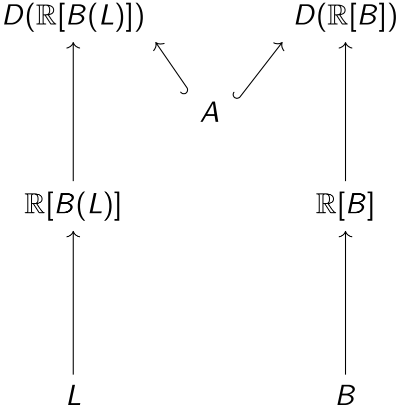
Also for this direction there are two ways.

- If $B \in$ **DeV**, then we can think of it as a boolean frame. We consider the algebra $\mathcal{C}^*(B)$ of the bounded elements of $\mathcal{C}(B)$.
- We lift $\prec$ of B to $\mathcal{C}^*(B)$ and we obtain $A \in$ **ubaℓ** as the set of its reflexive elements.

Alternatively,

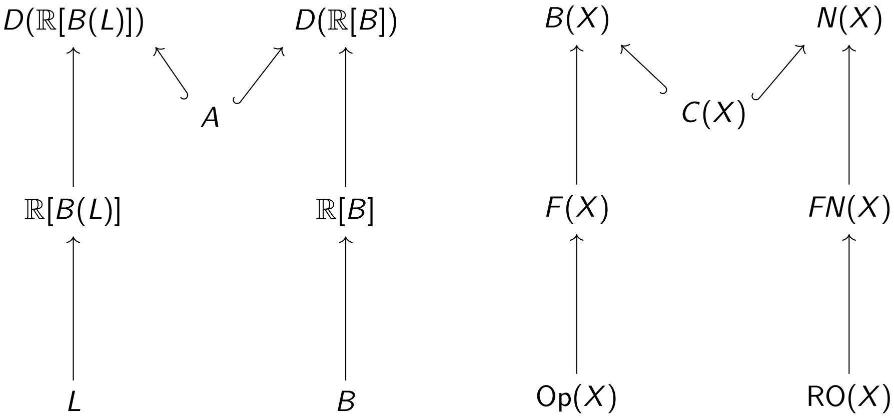
- Given $B \in$ **DeV**, we consider the Specker algebra $\mathbb{R}[B]$.
- Take its Dedekind completion $D(\mathbb{R}[B])$.
- We obtain $A \in$ **ubaℓ** as the set of reflexive elements of $D(\mathbb{R}[B])$ with respect to the proximity relation obtained by lifting $\prec$ of B.

$(D(\mathbb{R}[B]), \prec)$ is isomorphic to $(\mathcal{C}^*(B), \prec)$.



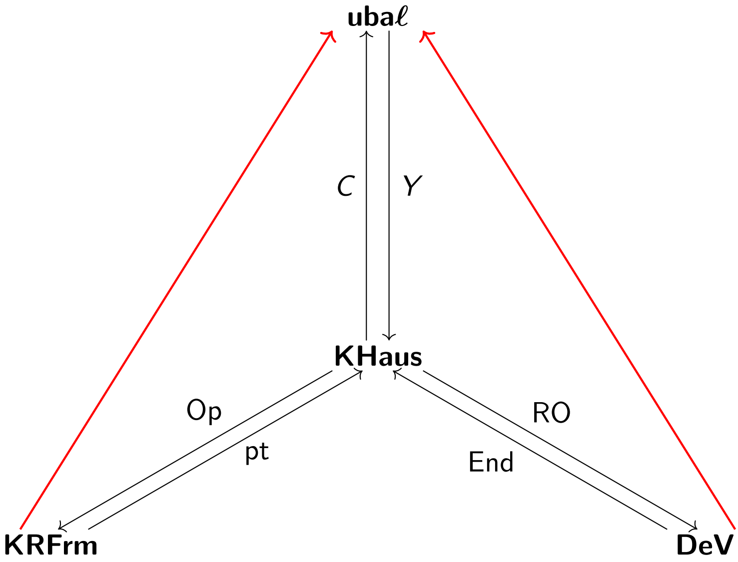
$D(\mathbb{R}[B(L)])$ is the canonical extension of A.

$D(\mathbb{R}[B])$ is the Dedekind completion of A.



$B(X)$ is the canonical extension of $C(X)$.

$N(X)$ is the Dedekind completion of $C(X)$.



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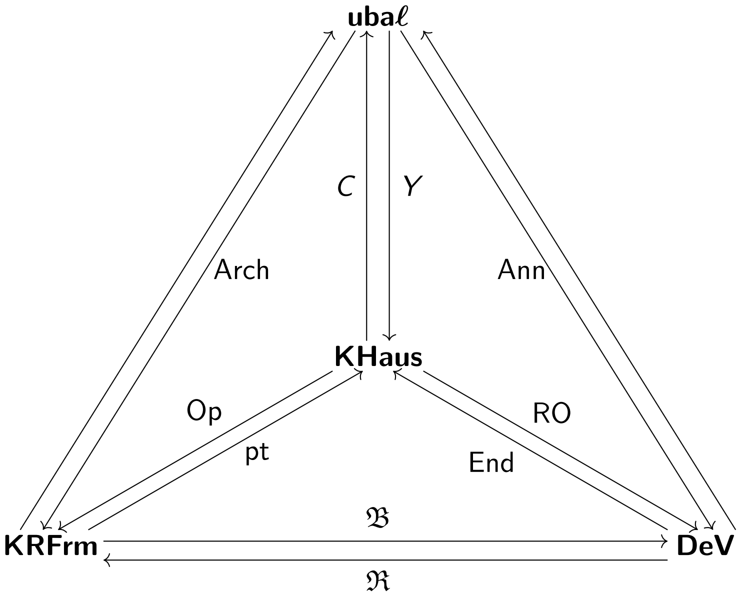

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THANK YOU!

DeV-morphisms

Definition

A de Vries homomorphism between de Vries algebras $(B, \prec)$ and $(C, \prec)$ is a map $h : B \to C$ satisfying

- $h(0) = 0$,
- $h(a \wedge b) = h(a) \wedge h(b)$,
- if $a \prec b$, then $\neg h(\neg a) \prec h(b)$,
- $h(a) = \bigvee\{h(b) \mid b \prec a\}$.

We denote by **DeV** the category of de Vries algebras and de Vries homomorphisms.