POLYHEDRAL SEMANTICS FOR MODAL LOGIC

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DOCToR: DUALITY, ORDER, (CO)ALGEBRAS, TOPOLOGY, AND RELATED TOPICS

VARIOUS TOPOLOGICAL SEMANTICS

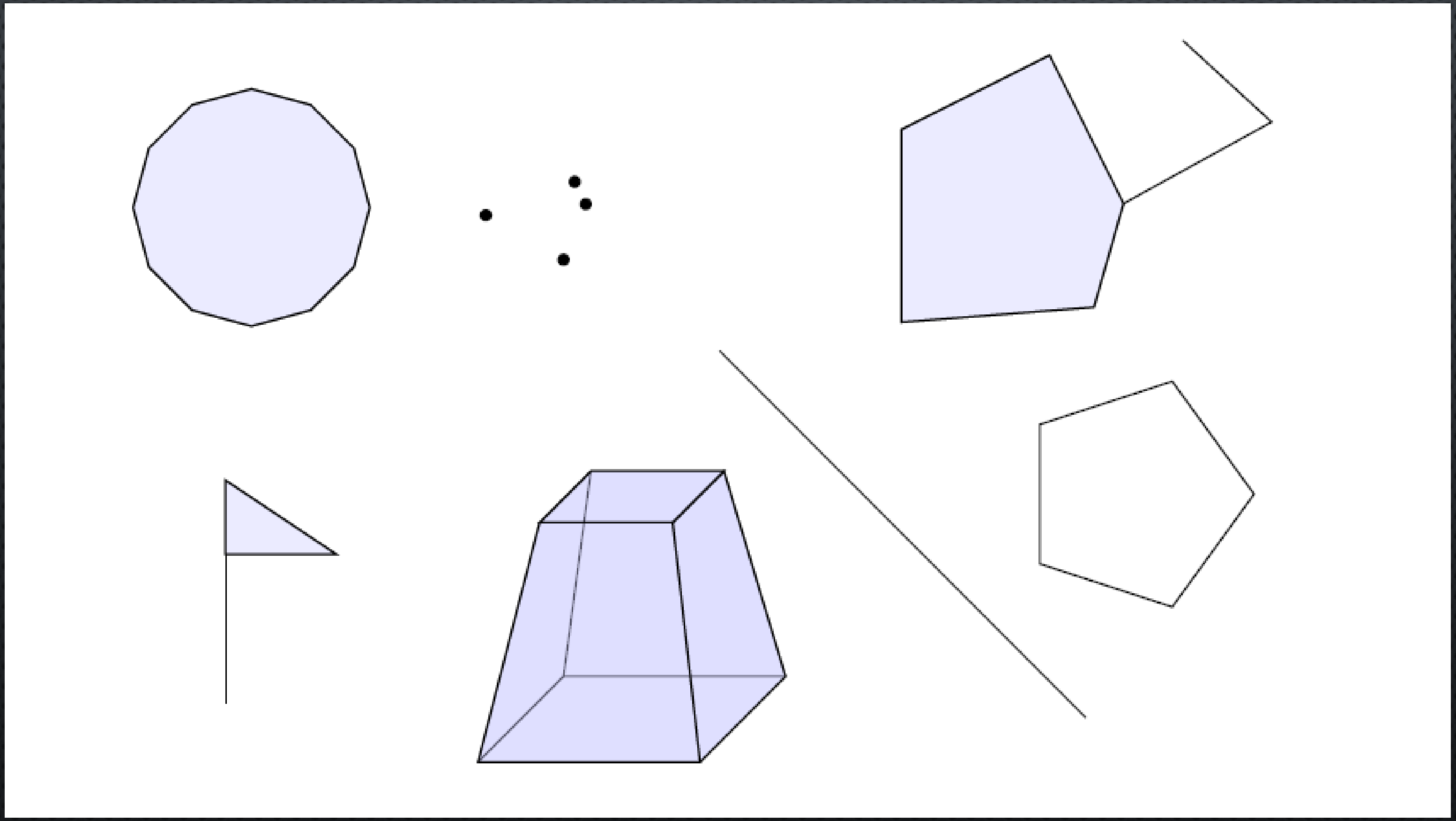
- A topological space X gives rise to:

Heyting algebra	Closure algebra	Derivative algebra
$Op(X)$	$(\wp(X), \boldsymbol{c})$	$(\wp(X), \boldsymbol{d})$
IPC	S4	wK4

- The algebra $Op(\mathbb{R}^n)$ generates the whole variety
- Subalgebras of $Op(\mathbb{R}^n)$ generate all subvarieties

Piecewise Linear Subsets

- Consider only polyhedral subsets of $\mathbb{R}^n$
- Generated by convex hulls of finite sets (polytopes)



Polyhedral algebras

- For a polyhedron P let $Sub(P)$ denote the closure algebra of subpolyhedra of P
- Call a modal logic (variety) polyhedral iff it is generated by polyhedral algebras

Questions:

- Which logics are polyhedral?
- What is a logic of a particular class of polyhedra?

OVERVIEW OF THE TALK

1. CRITERION FOR POLY-COMPLETENESS
2. POLY-COMPLETE AND POLY-INCOMPLETE LOGICS
3. STARLIKE LOGICS
4. CONVEX LOGICS
   based on joint work with S. Adam-Day, N. Bezhanishvili and E. Marra
5. FLAT POLYGONAL LOGICS
   based on joint work with K. Gogoladze, E. Kuznetsov, M. Jibladze, K. Razmadze and L. Uridia (Tbilisi)
6. APPLICATIONS TO SPATIAL MODEL CHECKING
   based on joint work with N. Bezhanishvili, V. Ciancia, G. Grilletti, D. Latella and M. Massink (CNR Pisa)

LOCAL FINITENESS

FOR POLYHEDRA A AND B: IF A ⍰ $B = \varnothing$ AND $A \subseteq \boldsymbol{c}B$

THEN $dim(A) < dim(B)$

- THIS IMPLIES TWO THINGS FOR THE ALGEBRA $Sub(P)$:
  - $Sub(P)$ IS OF FINITE HEIGHT, HENCE LOCALLY FINITE
  - $Sub(P)$ IS A **S4.GRZ-ALGEBRA**
- THUS, FINITE SUBALGEBRAS DETERMINE EVERYTHING

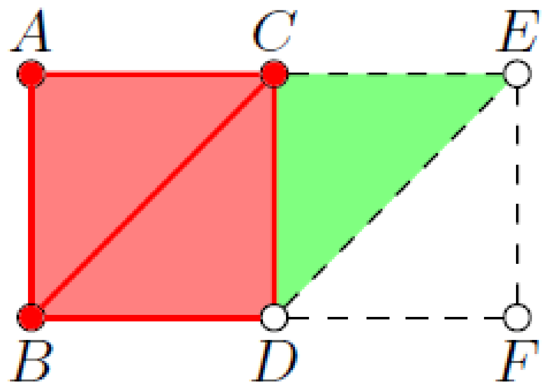
THEOREM: EACH POLY-COMPLETE LOGIC HAS FMP.

Finite subalgebras of $Sub(P)$

- Take $P_1, P_2, \dots, P_n \subseteq P$ and generate a closure subalgebra S inside $Sub(P)$
- This produces a subdivision Σ of P (finite partition into subpolyhedra $\Sigma = At(S)$)
- The subalgebra S is dual to a finite poset $(\Sigma, \preccurlyeq)$:
  - Take Σ to be atoms of S
  - Order by taking $A \preccurlyeq B$ iff $A \subseteq \mathbf{c}B$
- There is a natural interior map $f: P \to \Sigma$

ILLUSTRATION

ILLUSTRATION



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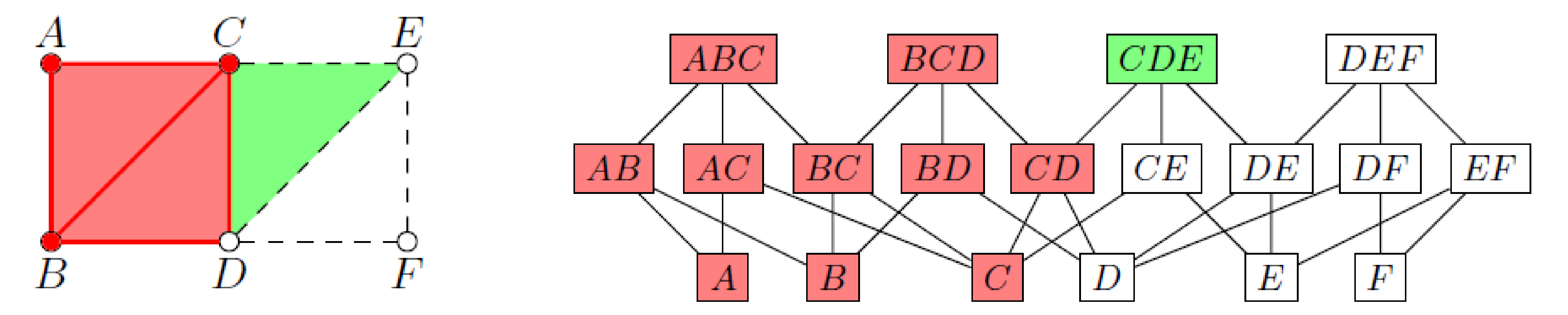
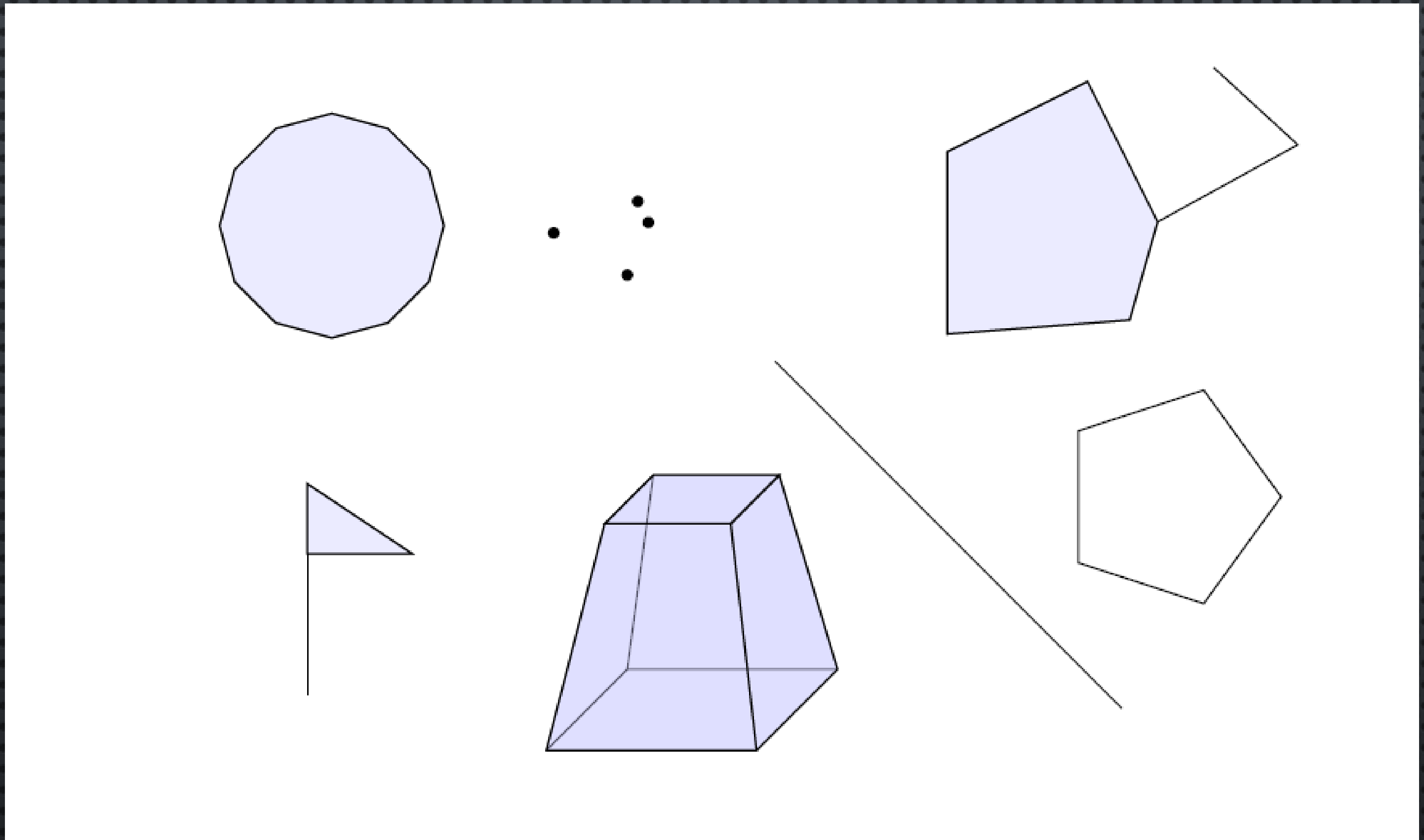
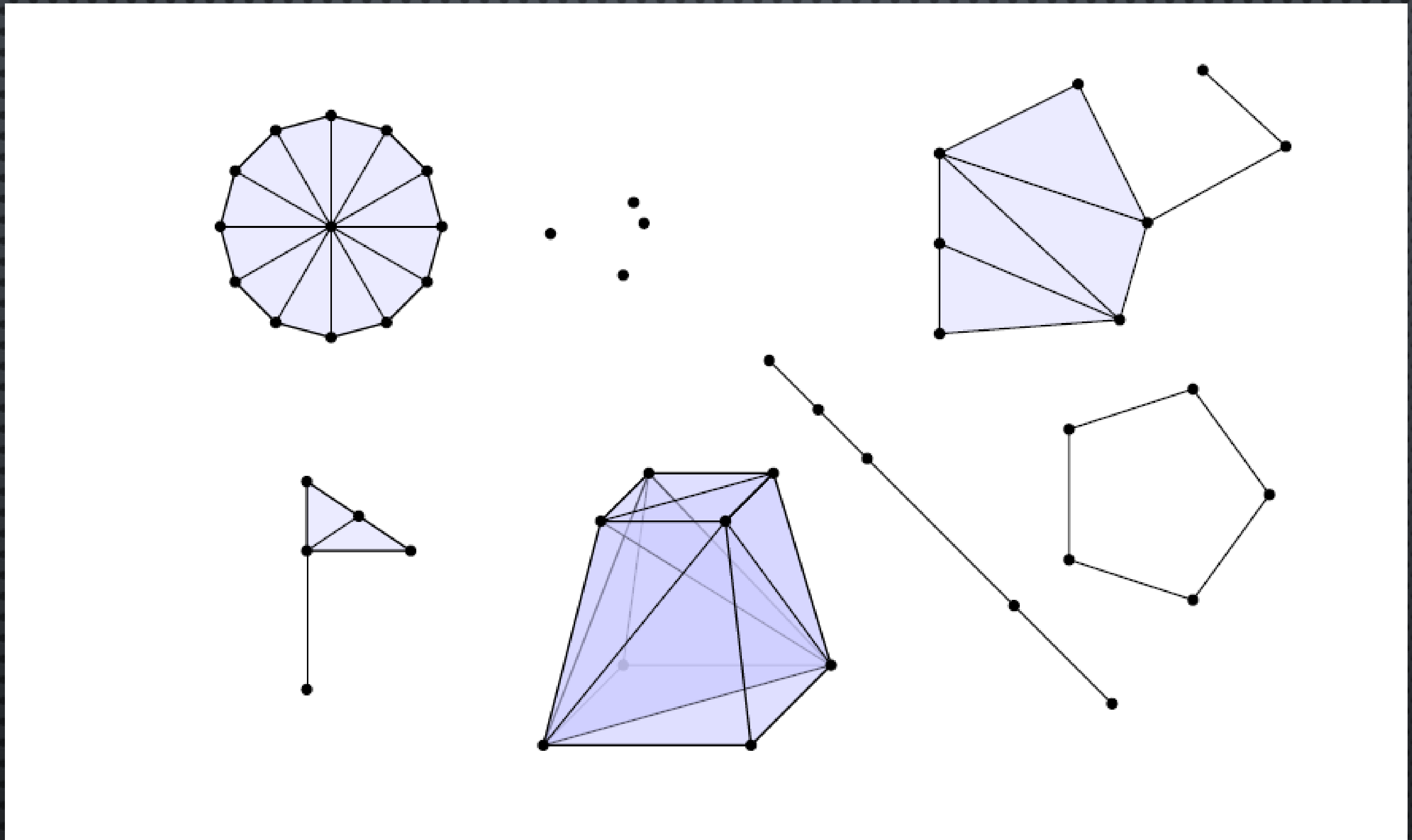


TRIANGULATIONS



- INTUITION: SUBDIVIDE POLYHEDRA INTO PIECES OF SIMPLEST POSSIBLE SHAPE

TRIANGULATIONS



- The pieces are points, (open) segments, (open) triangles, (open) tetrahedra, etc. – (open) simplices
- Every polyhedron admits a triangulation (simplicial complex*)

Triangulation subalgebras

Definition: Given a triangulation Σ of P, let $P(\Sigma)$ be the (closure) subalgebra generated by Σ inside $Sub(P)$.

Lemma: Each finite subalgebra of $Sub(P)$ is a finite subalgebra of a triangulation subalgebra.

Theorem: The logic of a polyhedron is the logic of its triangulations.

Proof: A non-theorem is refutable in a finite subalgebra. By the Lemma, in a triangulation subalgebra.

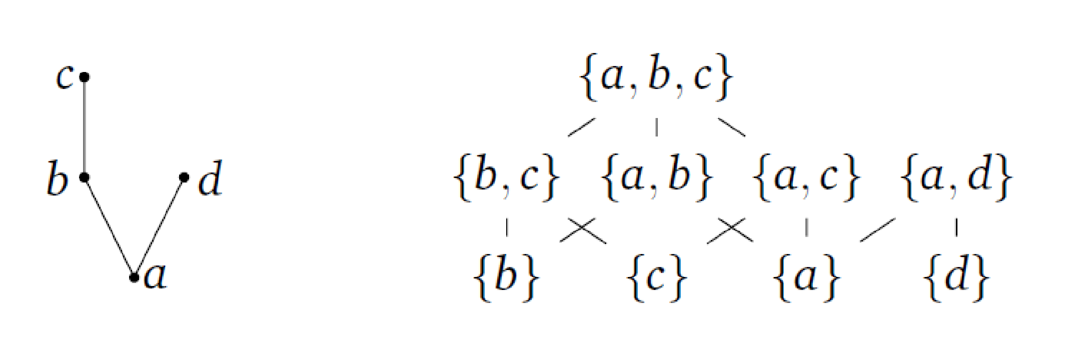
FROM POSETS TO POLYHEDRA

- EACH SUBDIVISION Σ OF P PRODUCES AN INTERIOR MAP:

$$f : P \to \Sigma$$

QUESTION: GIVEN A FINITE POSET F, IS THERE A POLYHEDRON THAT INTERIOR-MAPS ONTO F?

Definition (Alexandrov's Nerve):
The nerve $N(F)$ of a finite poset F is the set of non-empty chains of F ordered by inclusion



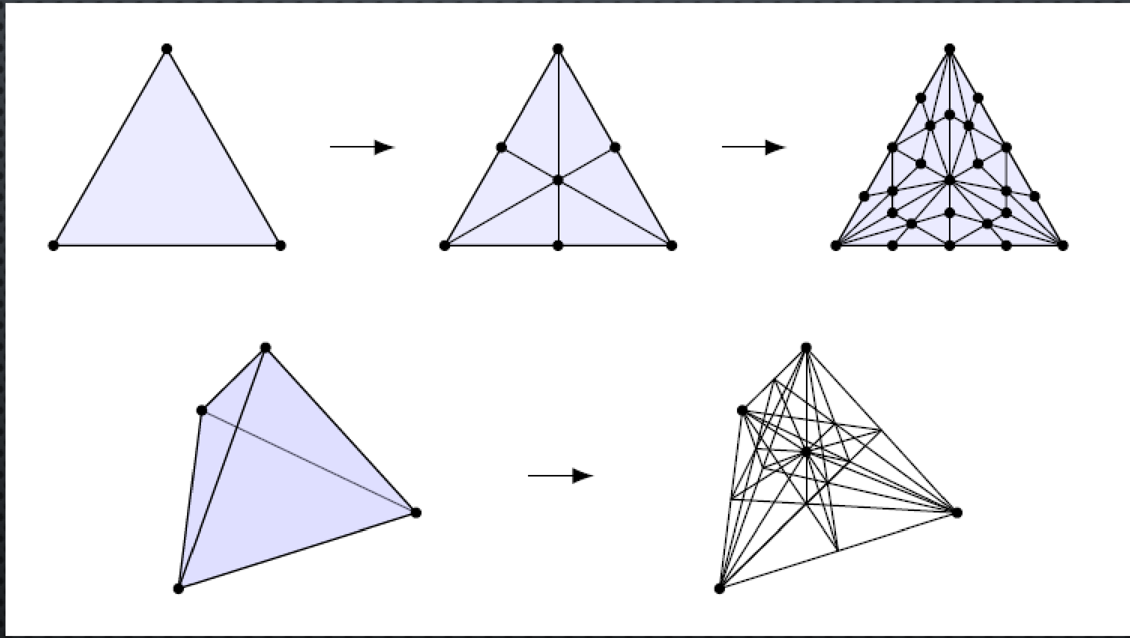
Theorem:
1) The nerve of F maps p-morphically (=interior map) onto F by:

$$\max : N(F) \to F$$

2) There exists a polyhedron P that interior-maps onto $N(F)$.

NERVES AND BARYCENTRIC SUBDIVISIONS

- FOR A TRIANGULATION Σ CONSTRUCT ITS BARYCENTRIC SUBDIVISION Σ' BY PUTTING A NEW POINT AT THE BARYCENTER OF EACH CELL AND FORMING A NEW TRIANGULATION AROUND IT



- THE POSETS Σ' AND $N(\Sigma)$ ARE ISOMORPHIC

The Nerve Criterion

Theorem (nerve criterion):

A logic L is poly-complete iff L is a logic of a class of finite frames closed under $N(\cdot)$

Intuition:

- Let P be a polyhedron and Σ its triangulation
- Let $\Sigma^{(n)}$ be $N^n(\Sigma)$ (the n^{th} iterated barycentric subdivision of Σ)
- Each finite subalgebra S of $Sub(P)$ is a subalgebra of some $\mathrm{P}(\Sigma^{(n)})$
- $\{\mathrm{P}(\Sigma^{(n)}) \mid n < \omega\}$ approximate $Sub(P)$

Some Consequences

Corollary

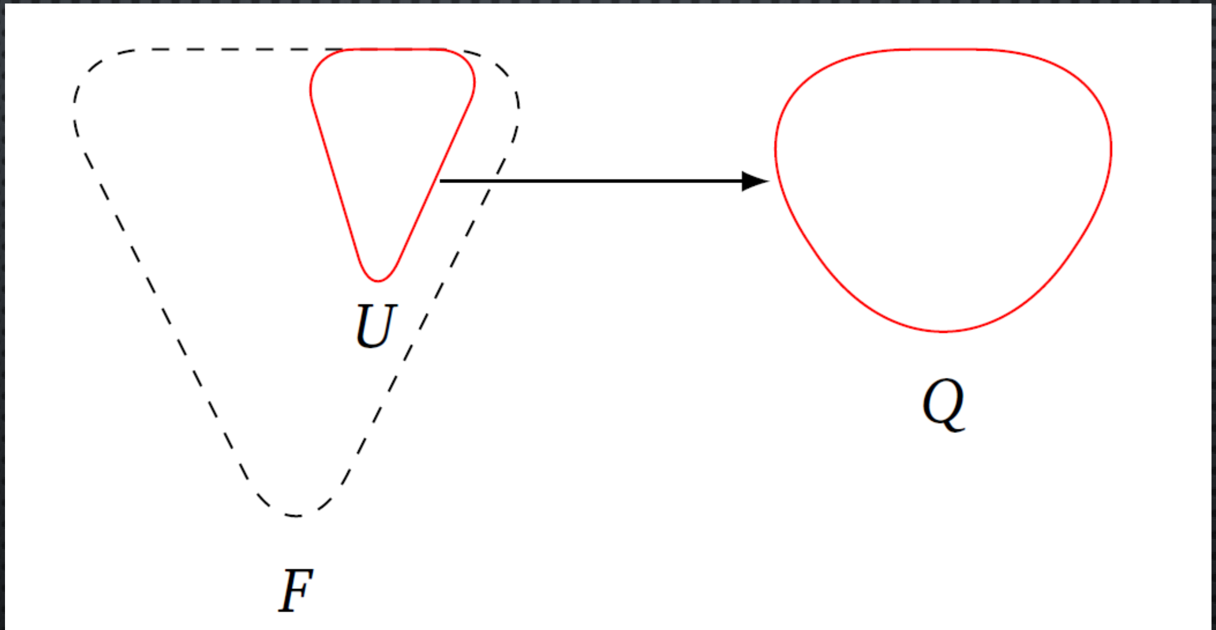
- The logics S4.Grz, S4.Grz$_n$ are poly-complete
- The logics S4.Grz.2, S4.Grz.3, S4.Grz$_n$.3 are poly-incomplete
- There are continuum many poly-incomplete logics with FMP (all stable logics)

Key intuitions:

- Use the Nerve Criterion and note that S4.Grz$_n$ is the logic of all posets of height $< n$ and the nerve construction does not increase the height
- The nerve contruction increases the width indefinitely

Jankov-Fine Formulas

- For each finite rooted frame Q there is a formula $\chi(Q)$, the Jankov-Fine formula of Q, such that for any frame F, $F \nvDash \chi(Q)$ iff F up-reduces to Q.

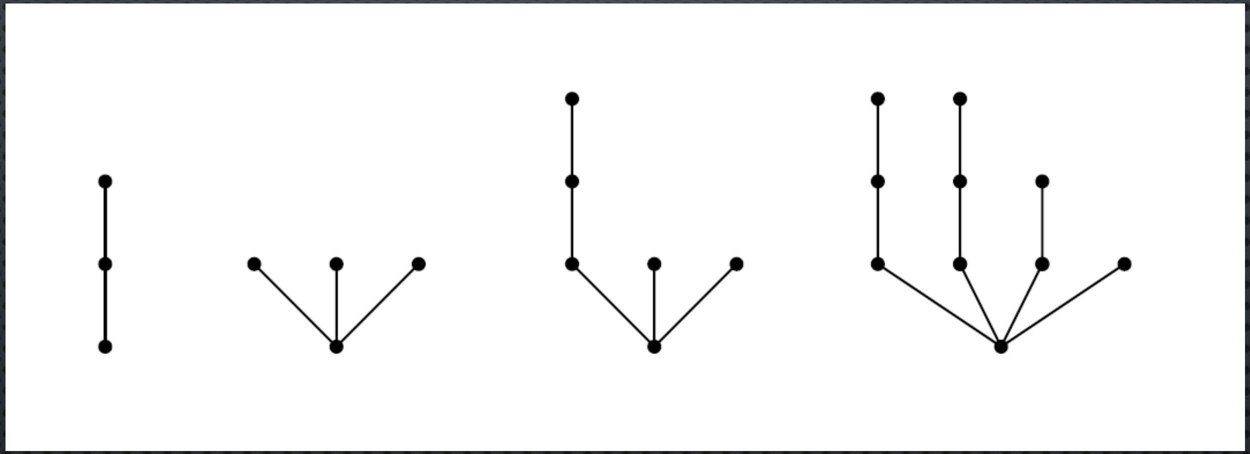


- The formula $\chi(Q)$ axiomatizes the logic that omits Q

STARLIKE TREES AND LOGICS

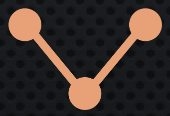
DEFINITION:

1) A TREE T IS STARLIKE IF THE ROOT IS THE ONLY BRANCHING NODE



2) A LOGIC L IS STARLIKE IF IT IS OF THE FORM

$$\mathrm{S4.Grz} + \chi(T_1) + \chi(T_2) + \cdots$$

WITH $\{T_1, T_2, \ldots\}$ A SET OF STARLIKE TREES OTHER THAN 

Starlike poly-completeness

Theorem:

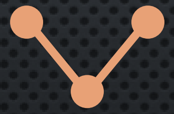
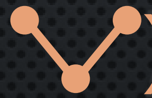
Every starlike logic is poly-complete

Corollary:

1) There are infinitely many poly-complete logics of each finite height

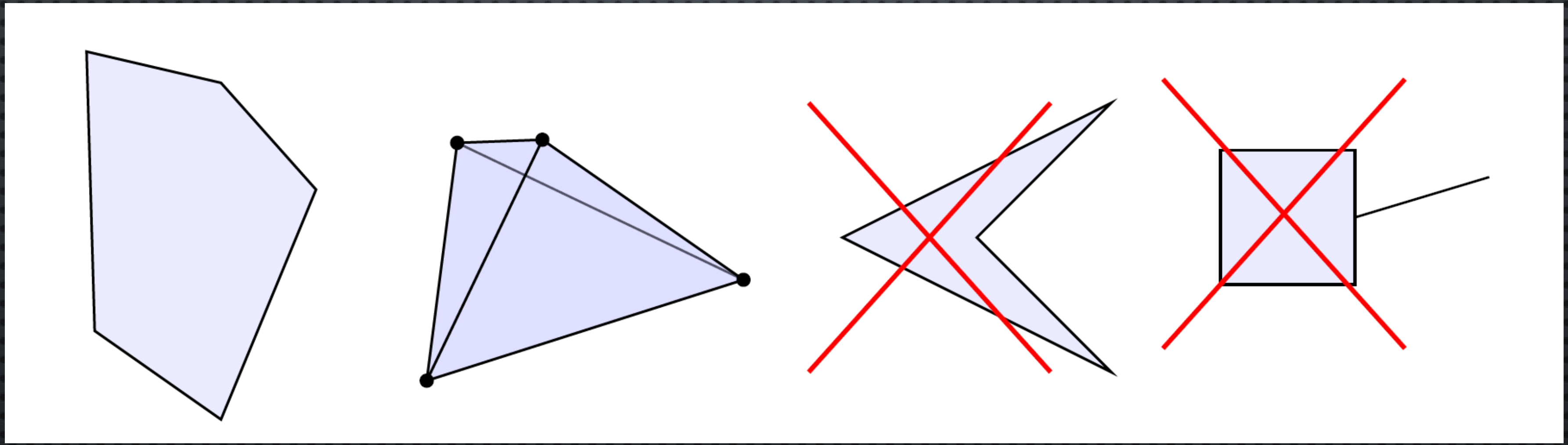
2) Scott's logic S4.Grz + $\chi($$)$ is poly-complete

STARLIKE POLY-COMPLETENESS

- STARLIKE LOGICS EXPRESS LOCAL CONNECTEDNESS PROPERTIES ABOUT FRAMES AND POLYHEDRA
- THERE ARE JUST COUNTABLY MANY STARLIKE LOGICS
- THE EXCLUSION OF  IS NECESSARY – THE ONLY POLY-COMPLETE LOGIC EXTENDING S4.GRZ + χ() IS TRIV

Convex Polyhedra

- The polyhedron P is convex if for any two points $x, y \in P$ it contains the straight segment $[x, y]$ connecting these points.

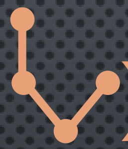


- The $\mathbb{R}^n$ is a paradigmatic example
- What is the logic of all convex polyhedra?

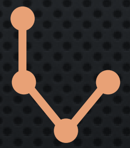
AXIOMATIZING CONVEX LOGICS

THEOREM:

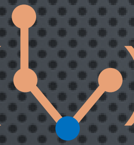
- The logic of convex polyhedra is axiomatized by

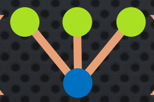
$\mathrm{PL} = \mathrm{S4.Grz} + \chi($$) + \chi($$)$

- The logic of convex polyhedra of dimension n, and the logic of $\mathbb{R}^n$ are both axiomatized by

$\mathrm{PL}_n = \mathrm{S4.Grz}_{n+1} + \chi($$) + \chi($$)$

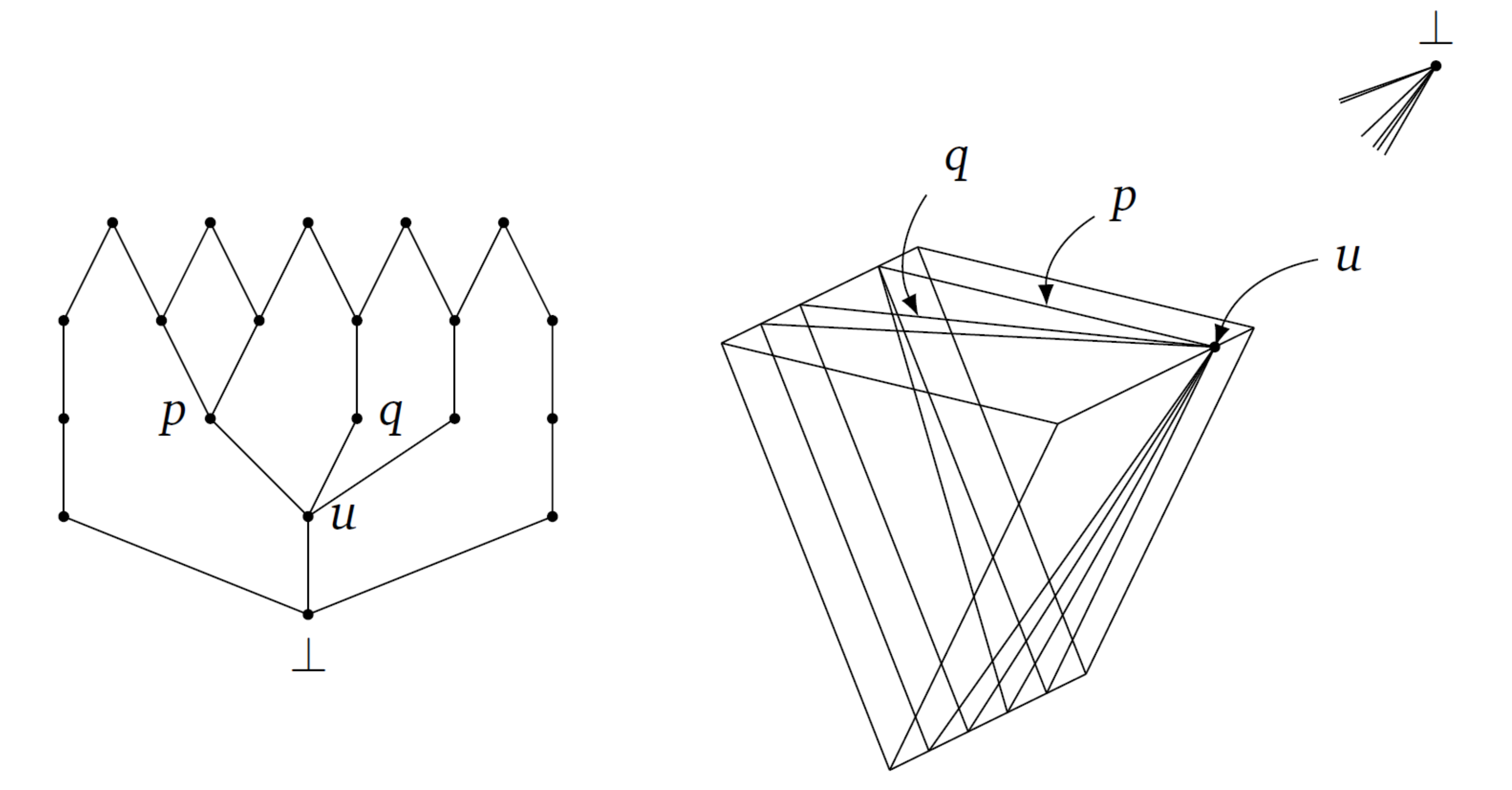
SOUNDNESS

- That $P \vDash \chi($$)$ expresses the classical result by Hurewicz and Wallman that a convex polyhedron of dimension *n* cannot be disconnected by a subset of dimension $< n - 1$

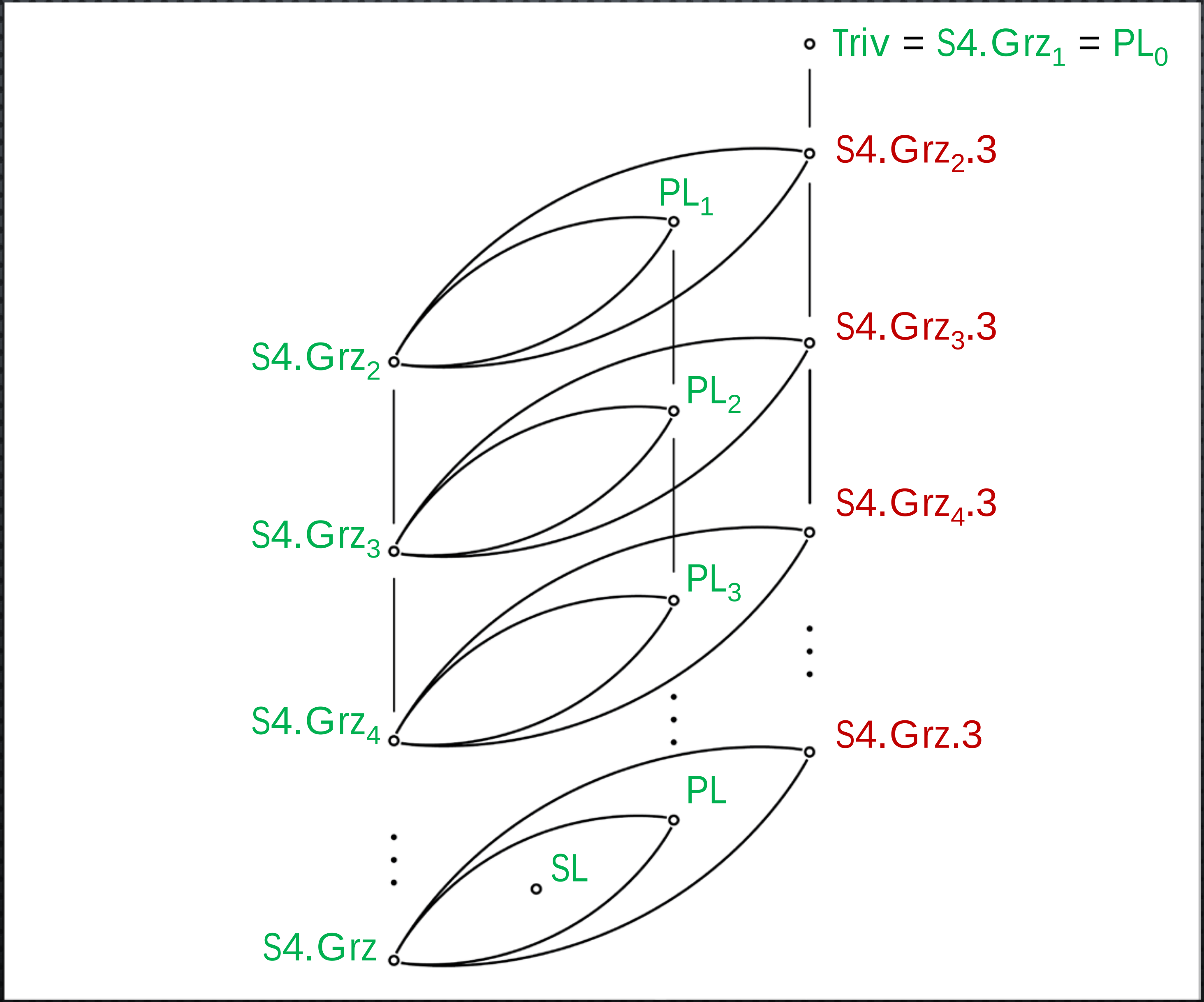
- That $P \vDash \chi($$)$ expresses that a convex polyhedron cannot contain three open disjoint subpolyhedra sharing a common boundary

COMPLETENESS

By transforming each appropriate finite frame into a saw-topped tree and then realizing it as a subdivision of a convex polyhedron



GENERAL PICTURE SO FAR

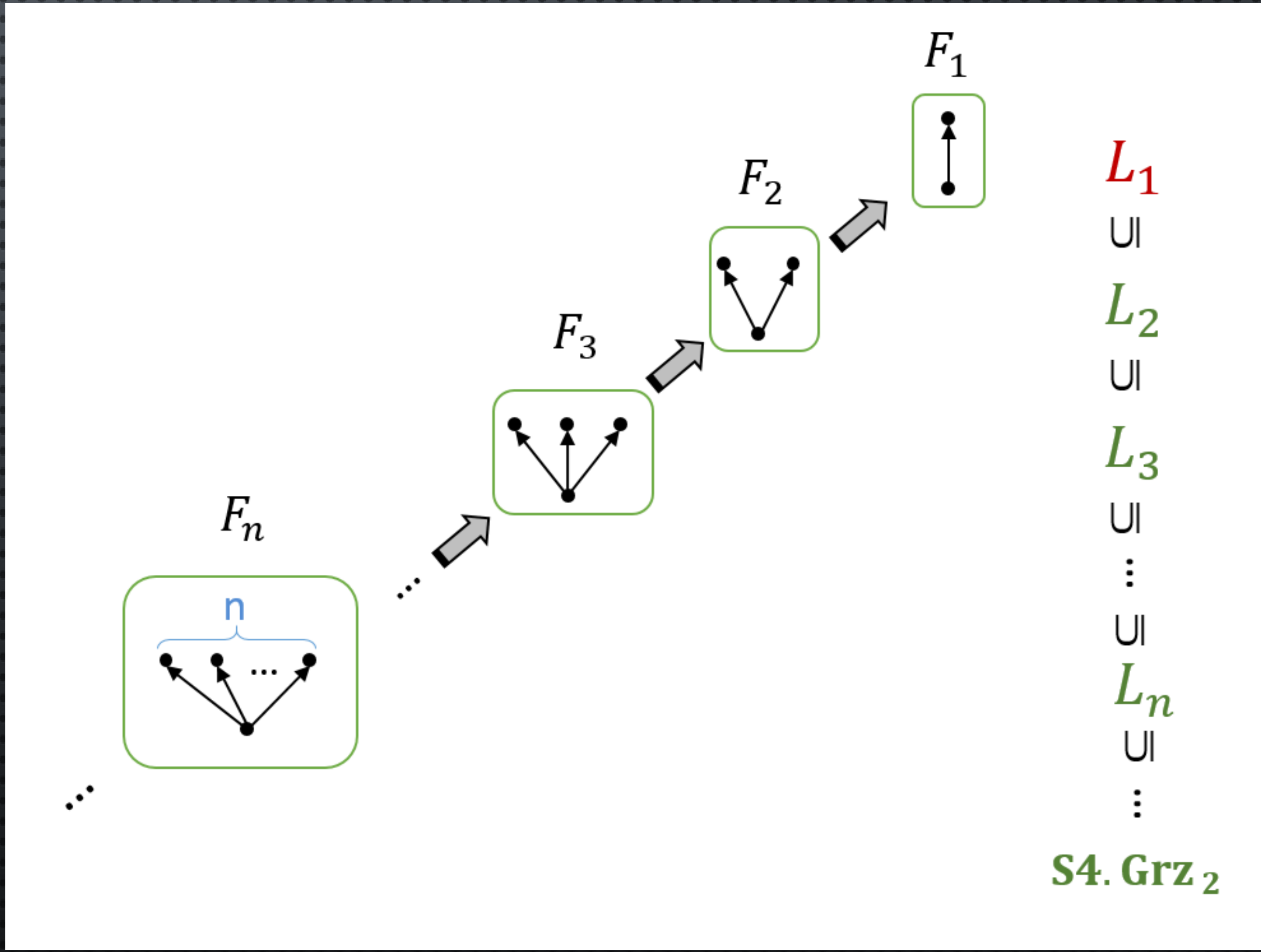


DIMENSION = 0

- The logic of the single point PL_0 = S4.Grz$_1$ = Triv

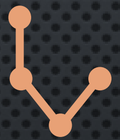
DIMENSION = 1

- EXTENSIONS OF S4.GRZ$_2$
- POSETS OF HEIGHT = 1
- $L_i = \mathrm{LOG}(F_i) = S4.Grz_2 + \chi(F_{i+1})$
- L_1 IS POLY-INCOMPLETE
- $L_2 = PL_2$

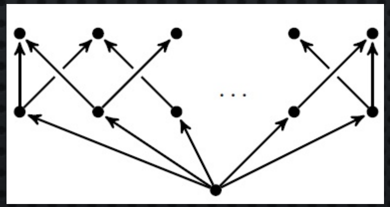


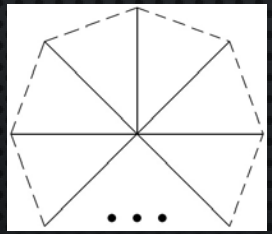
Dimension = 2

- Extensions of $\mathrm{S4.Grz}_3$ = the logic of all polygons
- The largest poly-complete logic is PL_2 =

In terms of forbidden frames: $\mathrm{S4.Grz}_{n+1} + \chi($  $) + \chi($  $)$

In terms of admitted frames:

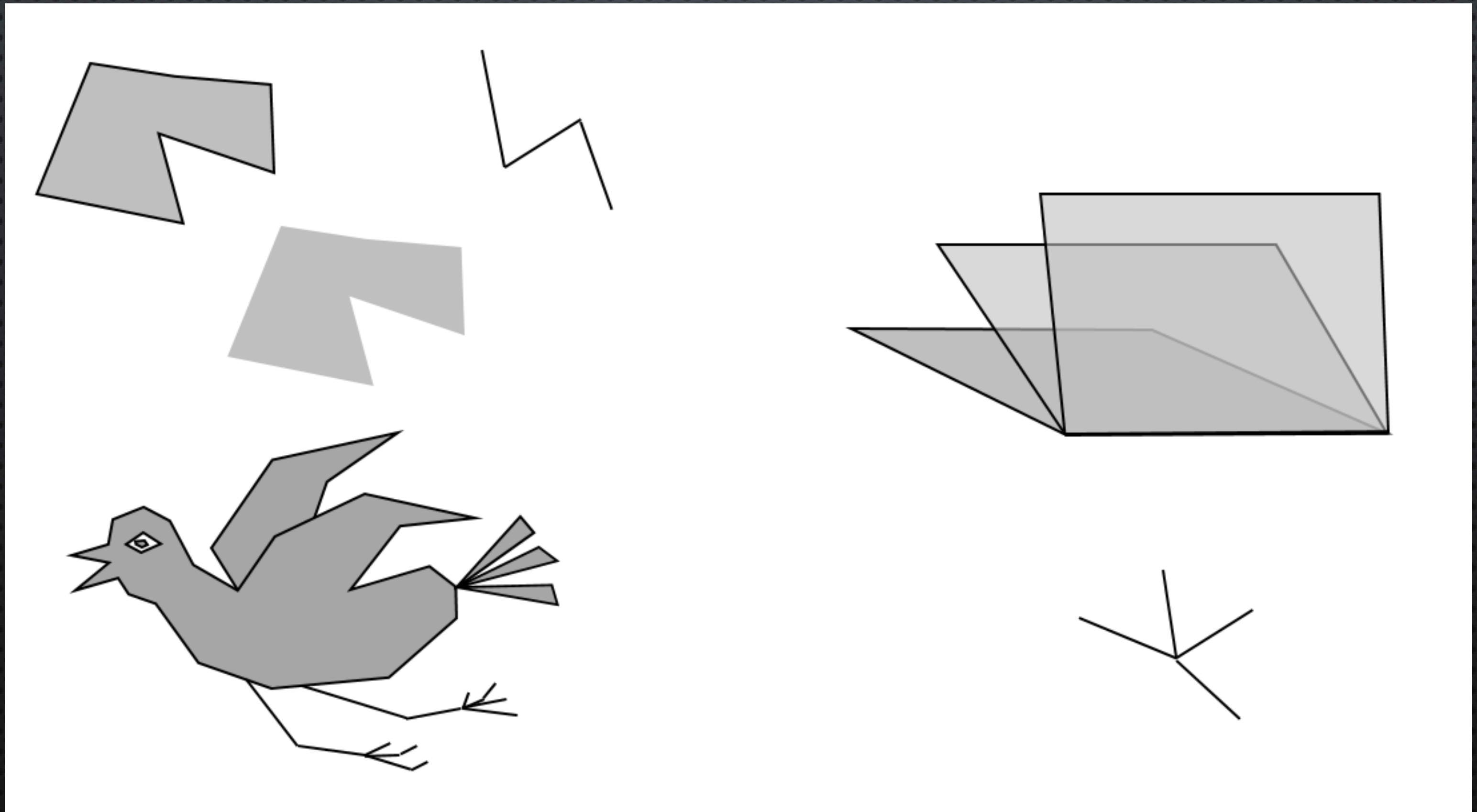




FLAT POLYHEDRAL LOGICS

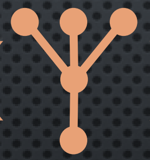
- AN n-DIMENSIONAL POLYHEDRON P IS CALLED FLAT IF IT IS EMBEDDABLE INTO $\mathbb{R}^n$

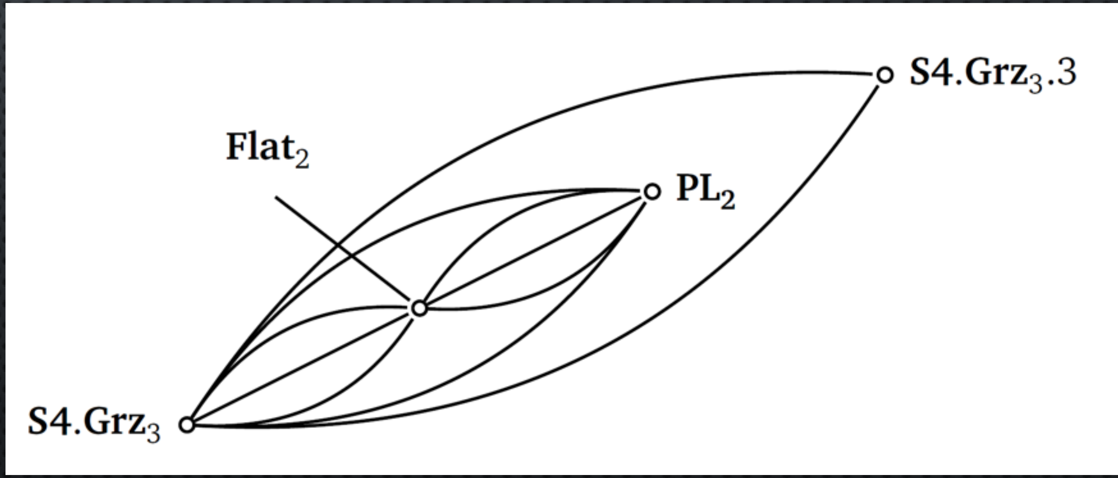
Flat Non-flat



THE LOGIC OF FLAT POLYGONS

- POLYGONAL SUBSETS OF THE 2D PLANE
- WHICH FINITE POSETS CAN WE OBTAIN FROM FLATS?
- WHICH FINITE POSETS CANNOT BE OBTAINED FROM FLATS?

$\mathrm{FLAT}_2 = \mathrm{S4.GRZ}_3 + \sigma($  $)$

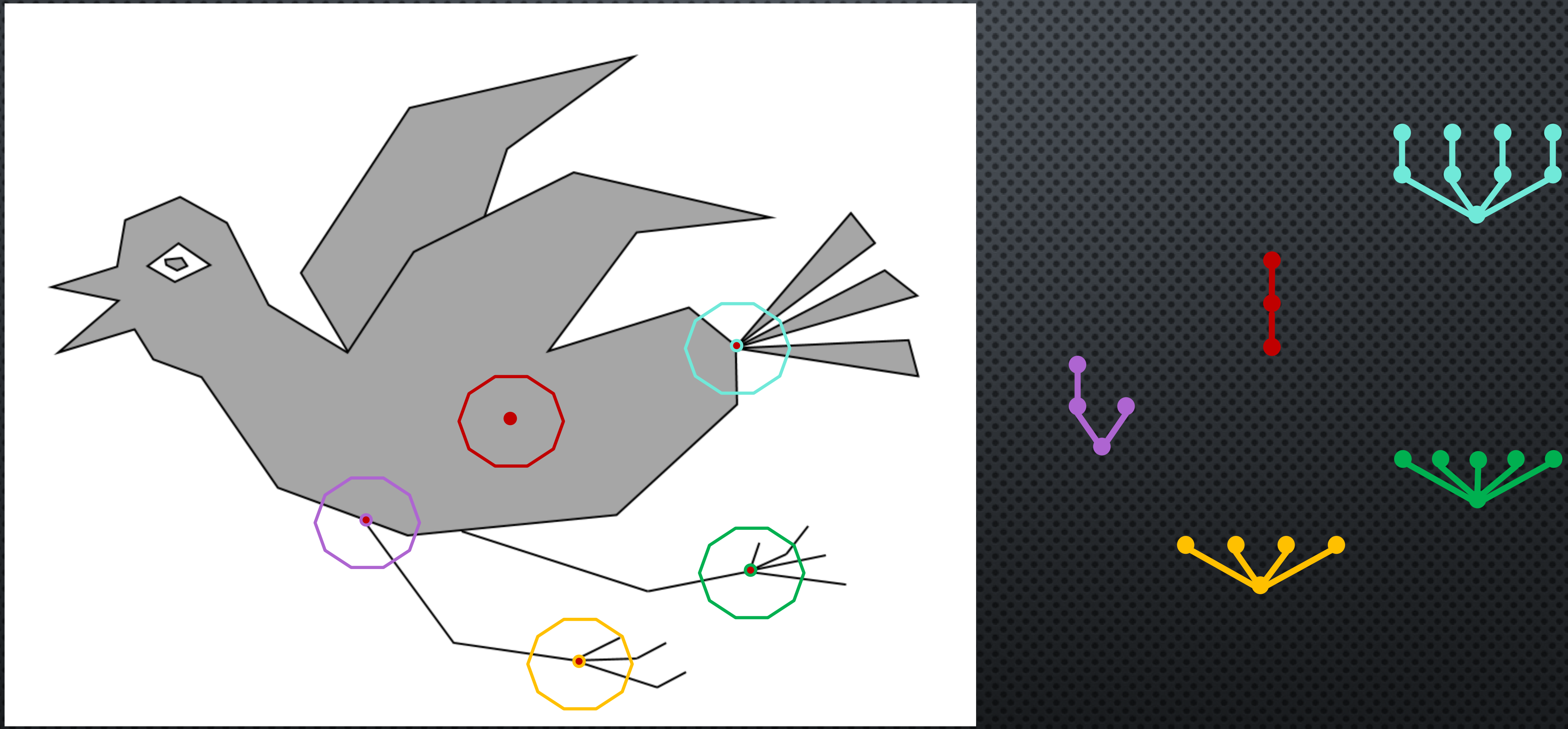


FLAT$_2$ IS A SUBFRAMIZATION OF PL$_2$

FLAT POLYGONAL LOGICS

- Take any collection of flat polygons, generate the modal logic. Call such logics flat polygonal
- Can we characterize flat polygonal logics?
- Let us take a single flat polygon and see which posets we can get

BIRD-OBTAINABLE POSETS



CHARACTERIZING FLAT POLYGONALS

Let α be an antichain in the ordering Q. Let L_α be the extension of S4.Grz$_3$ by $\chi(F)$, with $F \in \alpha$.

Lemma: Each α is finite.
Theorem: The flat polygonal logics are precisely the L_α.
Corollary: There are countably many flat polygonal logics, each finitely axiomatizable.

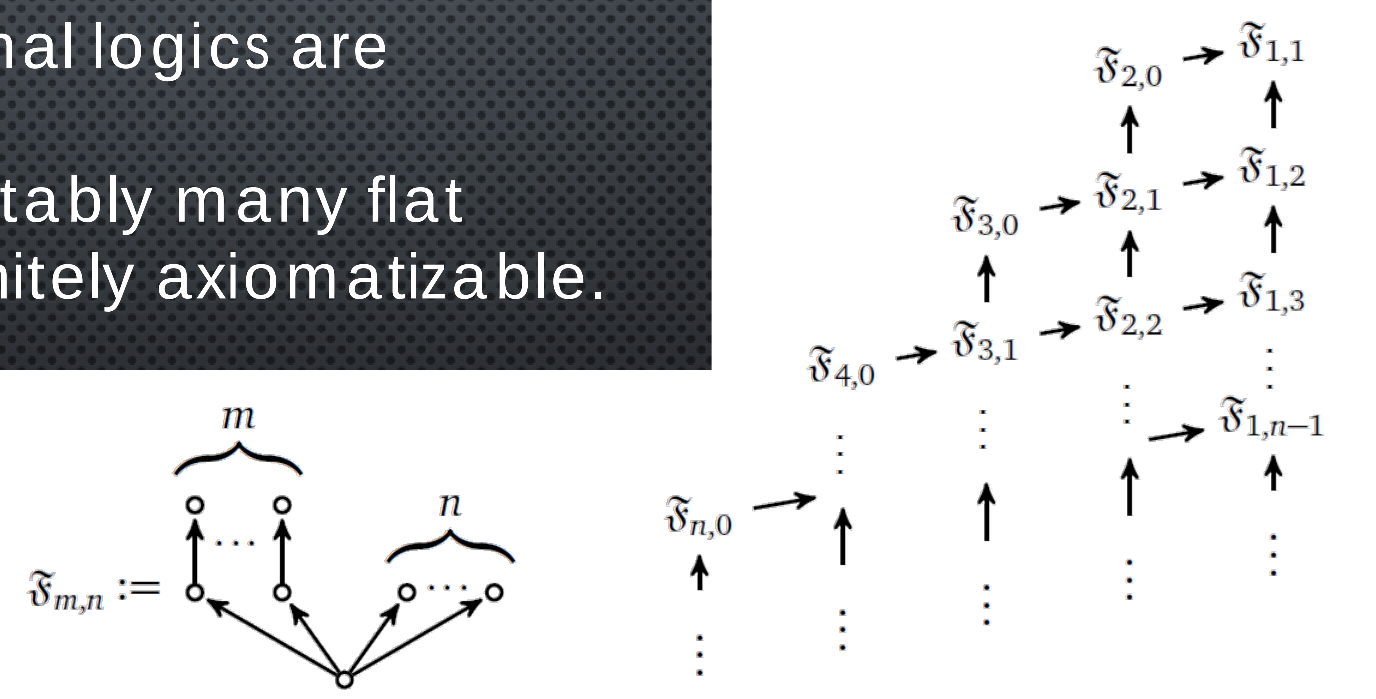


Figure 2: Poset Q of the frames $\mathfrak{F}_{m,n}$ ordered by reducibility

Still work to do in dim = 2 and up

Open questions in Dimension 2:

- Characterize all polygonal logics. Are there countably many?

Open questions in all dimensions:

- Characterize all flat logics
- Characterize all polyhedral logics

Applications

- Many real-world or fictional scenarios can be modelled using 3D polyhedra
- The theory developed so far is a groundwork for developing automated reasoning tools
- One such application is in the field of Spatial Model Checking
- The ultimate goal is to develop a tool to reason about 3D images (e.g. in Medical Imaging)

Reachability Operator

Definition:

A Path in P is a continuous map $\pi\colon [0,1] \to P$

Consider the binary modal operator $\gamma(\phi, \psi)$:

$$\mathcal{X}, x \vDash \gamma(\phi, \psi) \iff \textit{there exists a path } \pi \textit{ such that } \pi(0) = x,\ \pi(1) \in [\![\psi]\!]^{\mathcal{X}} \textit{ and } \pi((0,1)) \subseteq [\![\phi]\!]^{\mathcal{X}}$$

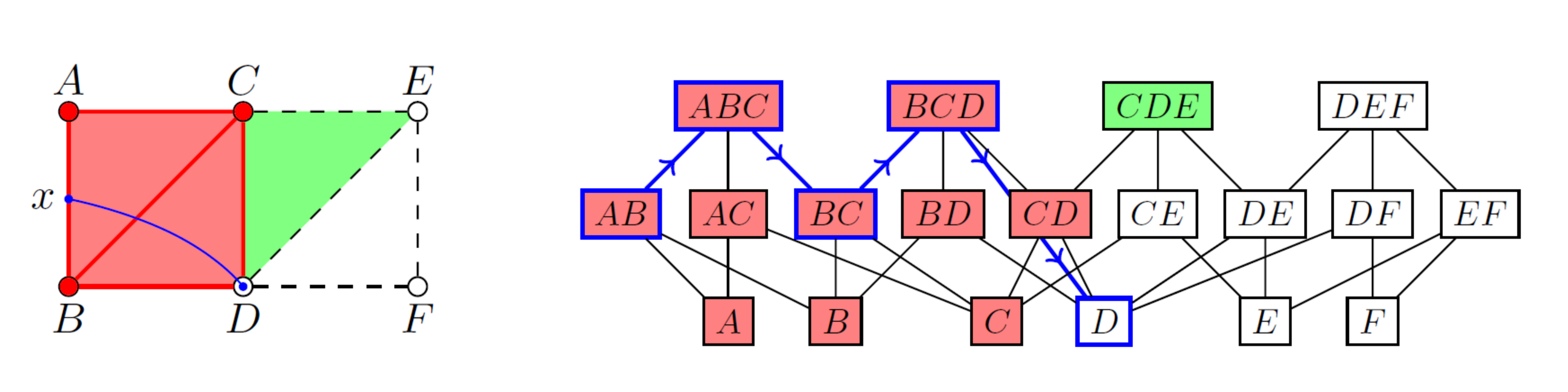
Expressing "ψ is reachable via a ϕ-path"

It turns out that $\gamma(A, B)$ is a polyhedron, whenever A and B are polyhedra.

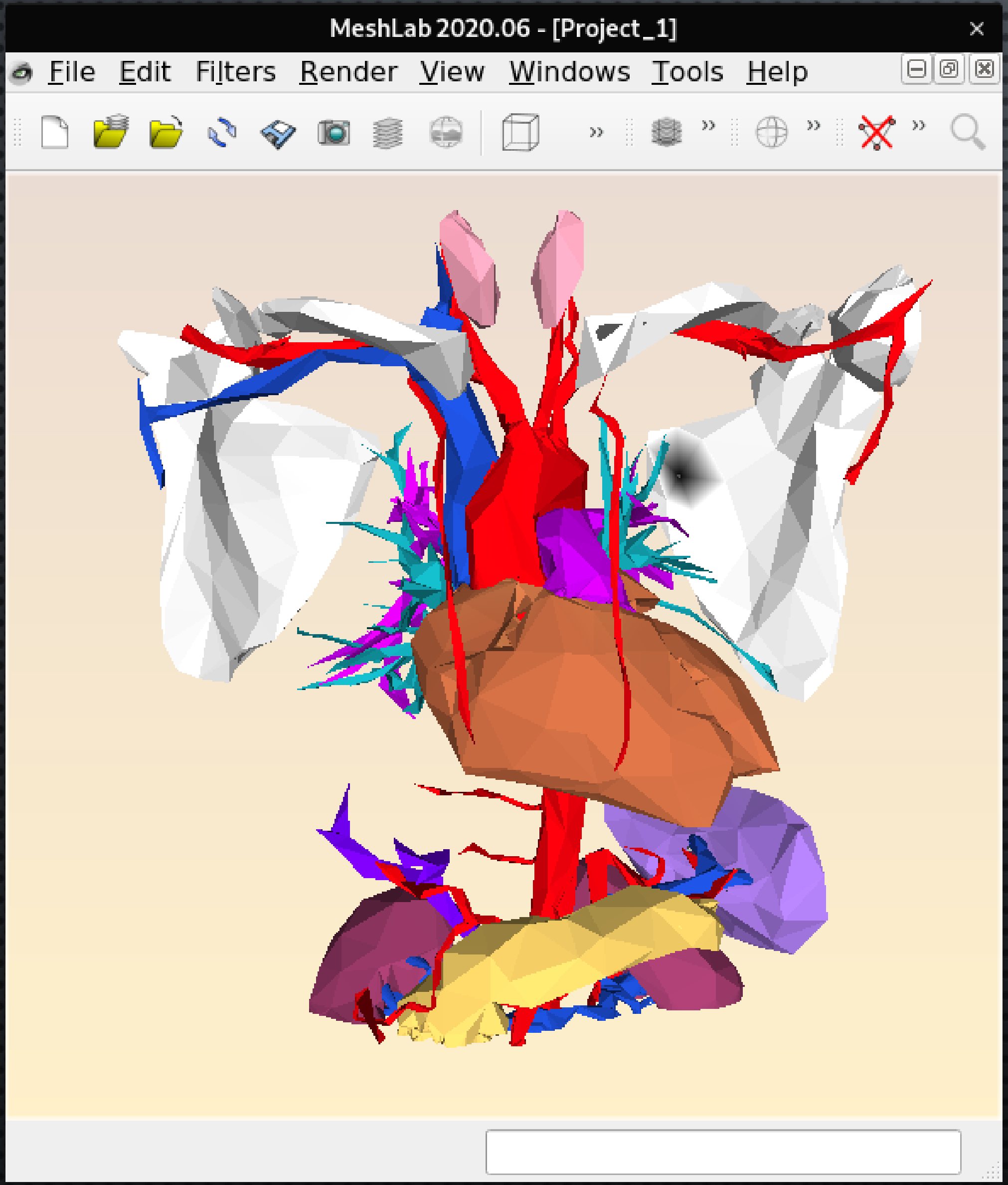
Passing down to the triangulation

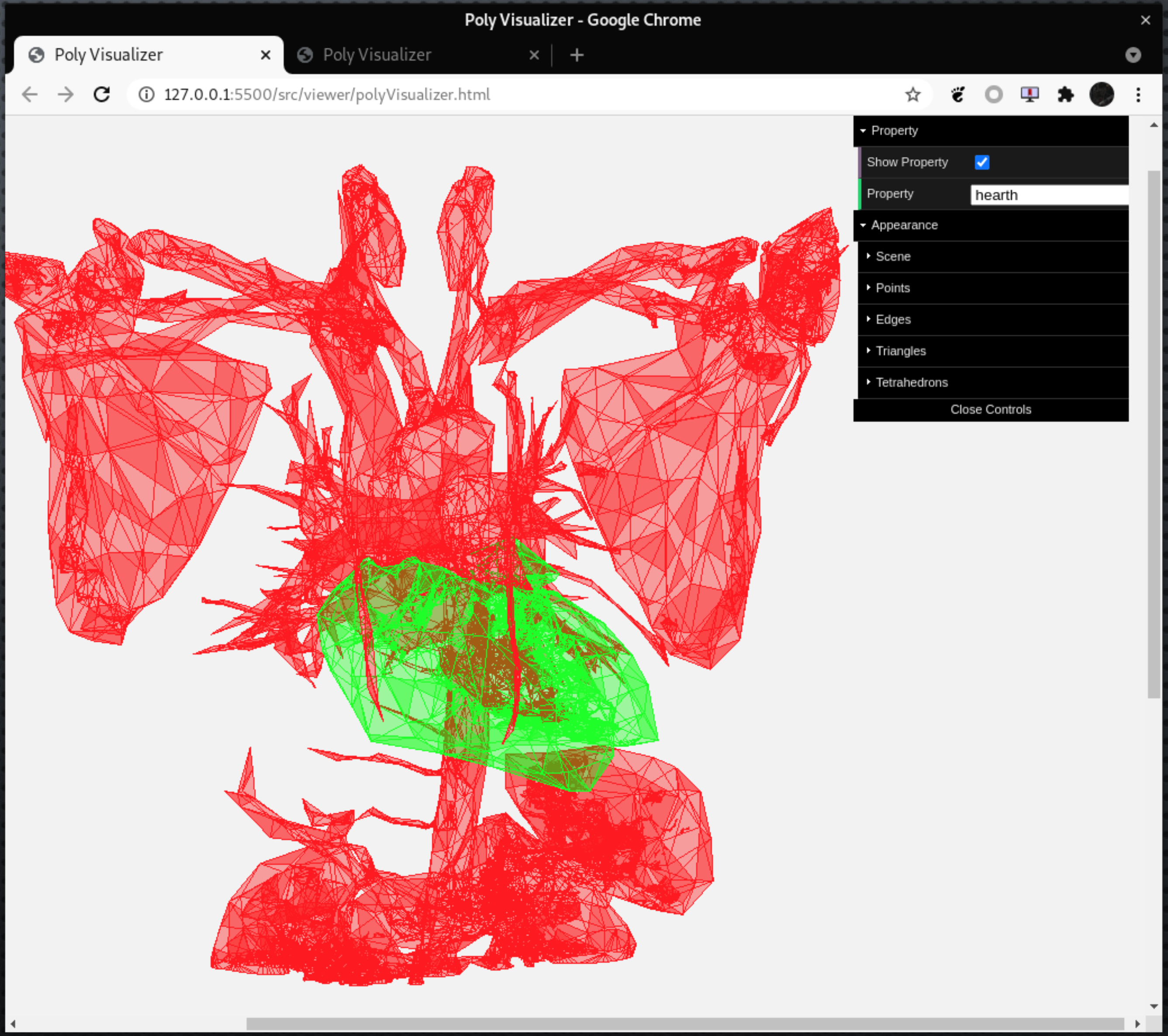
- Let Σ be a triangulation of a polyhedron P and let $f\colon P \to \Sigma$ be the corresponding map, then for each $x \in P$ and each formula ϕ in the reachability language:

$$P, x \vDash \phi \quad \text{iff} \quad \Sigma, f(x) \vDash \phi$$



PROTOTYPE TOOL/VISUALIZER





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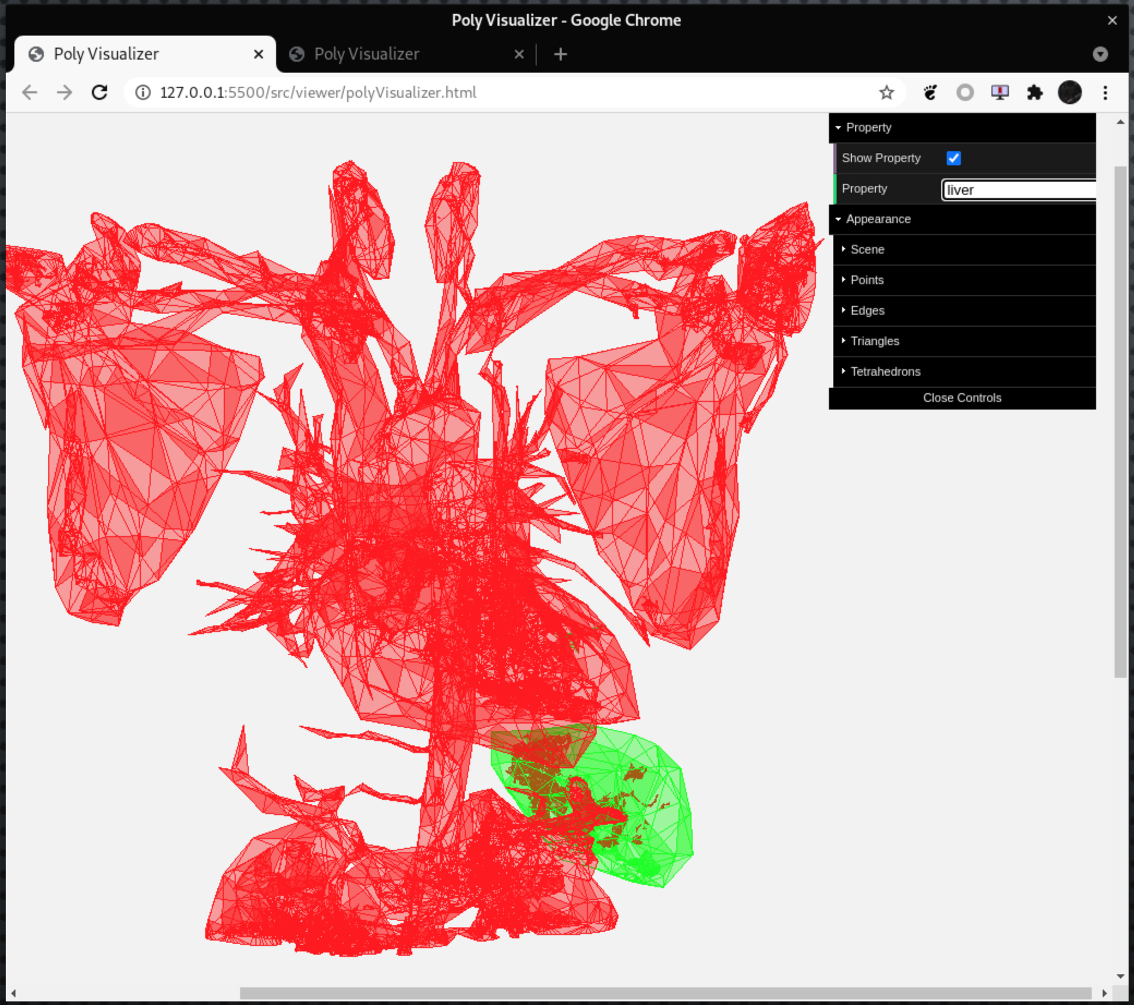
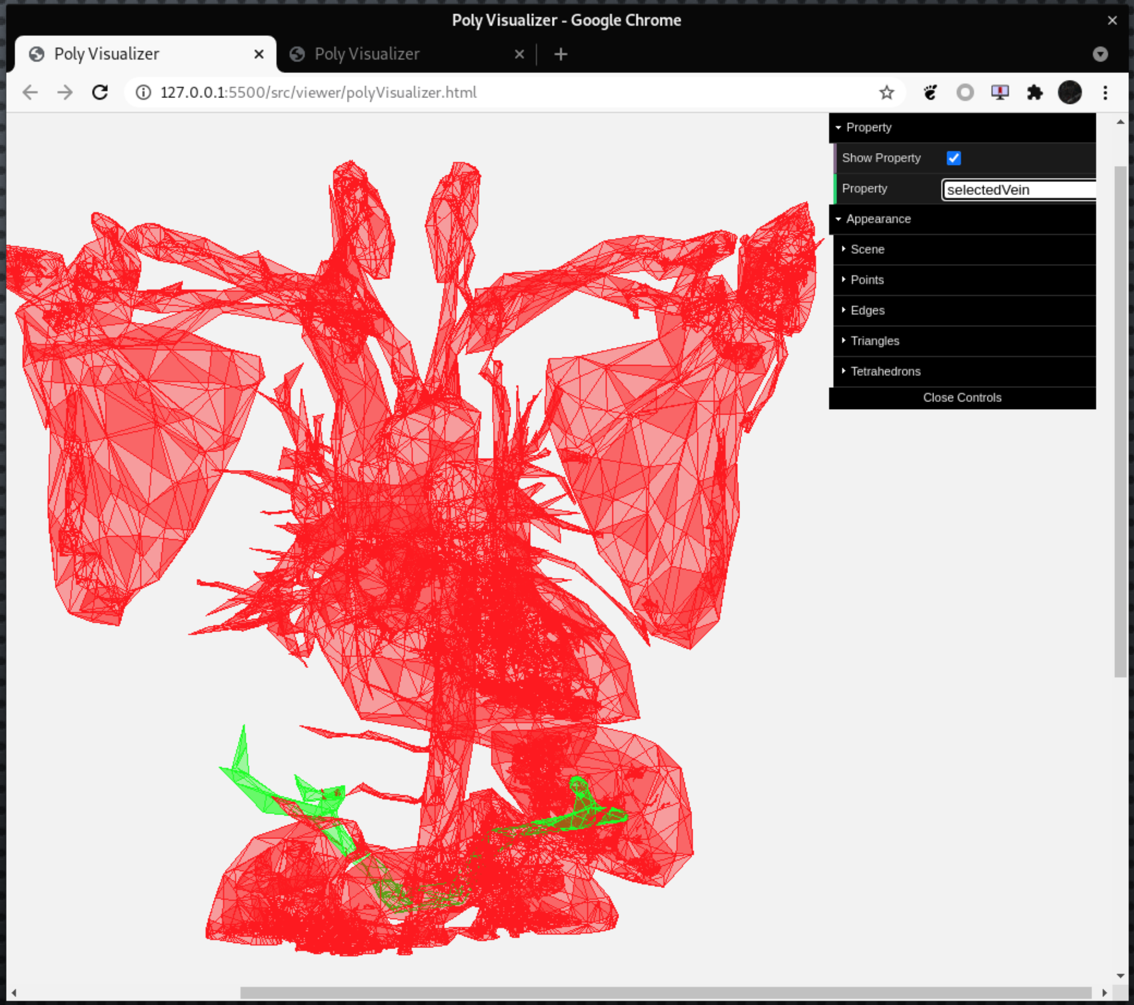


Liver selected



Veins reaching Liver selected

Some References

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4. The nerve criterion and polyhedral completeness of intermediate logics. Sam Adam-Day, Nick Bezhanishvili, David Gabelaia, Vincenzo Marra. *Submitted, September 2020.*

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