Combination of Uniform Interpolants via Beth Definability

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Outline

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1. Uniform Interpolants: the antefacts
2. Formal Preliminaries
3. Equality Interpolating Condition and Beth Definability
4. The Convex Combined Algorithm
5. The Non-Convex Case: a Counterexample
6. Conclusions

Uniform Interpolants in Propositional Logics

Uniform interpolants were introduced in the context of non-classical logics, starting from the pioneering work by Pitts [1992] who proved that in IPC for every formula $\phi(x, \underline{y})$ there is a formula $\phi_x(\underline{y})$ such that for every further formula $\psi(\underline{y}, \underline{z})$ we have

$$\phi(x, \underline{y}) \vdash \psi(\underline{y}, \underline{z}) \text{ iff } \phi_x(\underline{y}) \vdash \psi(\underline{y}, \underline{z})$$

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In modal logic, uniform interpolants have a *local* and a *global* version, depending on how the entailment $\vdash$ is interpreted.

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Uniform interpolants can be sematically connected to some appropriate notion of bisimulation at the level of Kripke models.

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The main results from the above literature are that uniform interpolants exist for intuitionistic logic and for some modal systems (like the Gödel-Löb system and the S4.Grz system); they do not exist for instance in S4 and K4, whereas for the basic modal system K they exist for the local version but not for the global version (the opposite situation is also well-possible, already in the locally tabular case).

Uniform Interpolants in First Order Theories

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Our interest in uniform interpolants for first order theories comes from infinite state model checking applications, in particular from the verification of (Business) Processes enriched with real data (*data-aware processes*).

Verification Applications

Given a state formula ϕ for states $S^{(i)}$, we symbolically define $T^{-1}(S^{(i)})$:

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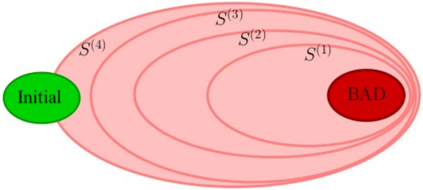
Backward-Reachability ($S^{(0)} \equiv$ "bad states")

Safety Check	If $S^{(i)}$ contains an initial, return **unsafe**
Next States	Compute $S^{(i+1)} := S^{(i)} \cup T^{-1}(S^{(i)})$
Fix-Point Check	If $S^{(i+1)} \equiv S^{(i)}$, return **safe**



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Then uniform interpolants enter into the picture. This might be competitive (as witnessed by our MCMT implementation) even from complexity viewpoint, because only a limited fragment of first-order logic is needed to formalize databases with primary and foreign keys. More details in our journal (MSCS 20) and conferences (BPM 19,20,21) papers.

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- We prove that the equality interpolating condition is also **necessary** for transferring covers.
- The algorithm relies on the extensive use of the **Beth definability property** for primitive fragments.
- **Counterexample** showing **non-transfer** of covers for non-convex theories in general, even in case combined quantifier-free interpolants do exist.

Outline

Preliminaries

Definition

Given a FO theory T and two quantifier-free FO formulae $\alpha(\underline{x}, \underline{y})$, $\beta(\underline{y}, \underline{z})$ such that $\vdash_T \alpha \to \beta$, a **quantifier-free** FO formula $\gamma(\underline{y})$ is a **T-quantifier-free interpolant** if $\vdash_T \alpha \to \gamma$ and $\vdash_T \gamma \to \beta$ hold.

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A theory T is *convex* iff for every **constraint** δ, if $T \vdash \delta \to \bigvee_{i=1}^{n} x_i = y_i$ then $T \vdash \delta \to x_i = y_i$ holds for some $i \in \{1, ..., n\}$.

A convex theory is 'almost' stably infinite.

Uniform Quantifier-Free Interpolation (Covers)

Fix a theory T and an existential formula $\exists \underline{e}\, \phi(\underline{e}, \underline{y})$.

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- In the cover $\psi(\underline{y})$, the variables $\underline{e}$ have been 'eliminated', in some sense.
- But, in general, $\psi(\underline{y})$ does *not* imply $\exists \underline{e}\, \phi(\underline{e}, \underline{y})$. Hence, usually $\psi(\underline{y})$ and $\exists \underline{e}\, \phi(\underline{e}, \underline{y})$ are not T-equivalent.

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Two further algorithms are in (Gianola, G., Kapur CILC 21).

Covers and Model Completions

A *universal* Σ-theory T has a ***model completion*** iff there is a stronger theory $T^* \supseteq T$ (in the same signature Σ) such that (i) every T-model embeds into a model of T^*; (ii) T^* eliminates quantifiers.

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Theorem (Covers and QE)

Suppose that T is a universal theory. Then, T has a model completion T^* ***iff*** *T has uniform quantifier-free interpolation. If this happens, T^* is axiomatized by the infinitely many sentences $\forall \underline{y}\, (\psi(\underline{y}) \to \exists \underline{e}\, \phi(\underline{e}, \underline{y}))$, where $\exists \underline{e}\, \phi(\underline{e}, \underline{y})$ is a primitive formula and ψ is a* ***cover*** *of it.*

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Hence, **computing covers** in a theory T
is **equivalent** to
eliminating quantifiers in its model completion T^*.

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Equality Interpolating Condition

Definition ([YM05])

A convex universal theory T is *equality interpolating* iff for every pair y_1, y_2 of variables and for every pair of *constraints* $\delta_1(\underline{x}, \underline{z}_1, y_1)$, $\delta_2(\underline{x}, \underline{z}_2, y_2)$ such that $T \vdash \delta_1(\underline{x}, \underline{z}_1, y_1) \wedge \delta_2(\underline{x}, \underline{z}_2, y_2) \to y_1 = y_2$, **there exists** a term $t(\underline{x})$ such that $T \vdash \delta_1(\underline{x}, \underline{z}_1, y_1) \wedge \delta_2(\underline{x}, \underline{z}_2, y_2) \to y_1 = t(\underline{x}) \wedge y_2 = t(\underline{x})$.

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Examples of universal **quantifier-free interpolating** and **equality interpolating** theories:

- $\mathcal{EUF}(\Sigma)$, given a signature Σ;
- recursive data theories;
- linear real arithmetics;
- Boolean algebras.

Beth Definability and Equality Interpolating Condition

Equality interpolating can be characterized using Beth definability.

Given a primitive formula $\exists \underline{z} \phi(\underline{x}, \underline{z}, y)$, we say that:

- $\exists \underline{z}\, \phi(\underline{x}, \underline{z}, y)$ *implicitly defines* y in T iff the following formula is T-valid: $\forall y\, \forall y'\, (\exists \underline{z} \phi(\underline{x}, \underline{z}, y) \wedge \exists \underline{z} \phi(\underline{x}, \underline{z}, y') \rightarrow y = y')$;
- $\exists \underline{z} \phi(\underline{x}, \underline{z}, y)$ *explicitly defines* y in T iff there is a term $t(\underline{x})$ such that the formula is T-valid: $\forall y\, (\exists \underline{z} \phi(\underline{x}, \underline{z}, y) \rightarrow y = t(\underline{x}))$;
- a theory T has the ***Beth definability property*** *for primitive formulae* iff whenever a primitive formula $\exists \underline{z}\, \phi(\underline{x}, \underline{z}, y)$ *implicitly* defines the variable y then it also *explicitly* defines it.

Theorem (Key Theorem [BGR14])

A convex theory T having quantifier-free interpolation is ***equality interpolating iff*** *it has the* ***Beth definability property*** *for primitive formulae.*

Strong Amalgamability and Equality Interpolating Condition

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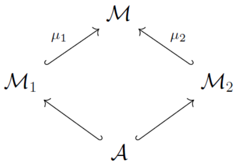
Theorem

*[BGR14] The following two conditions are equivalent for a convex universal theory T: (i) T is **equality interpolating** and has **quantifier-free interpolation**; (ii) T has the **strong amalgamation property**.*

Strong Amalgamability and Equality Interpolating Condition

Recall that a universal theory T has the *strong amalgamation property* iff every pair of models $\mathcal{M}_1, \mathcal{M}_2$ of T sharing a common submodel $\mathcal{A}$ can be amalgamated over $\mathcal{A}$ into a model $\mathcal{M}$ in such a way that the $\mathcal{A}$-embeddings μ_1, μ_2 satisfy the following additional condition: if for some m_1, m_2 we have $\mu_1(m_1) = \mu_2(m_2)$, then there exists an element a in $|\mathcal{A}|$ such that $m_1 = a = m_2$.



Transfer of Quantifier-free Interpolants

Theorem (Sufficient Condition [YM05, BGR14])

Let T_1 and T_2 be two universal, convex, stably infinite theories over disjoint signatures Σ_1 and Σ_2. If both T_1 and T_2 are equality interpolating and have quantifier-free interpolation property, then so does $T_1 \cup T_2$.

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There is a converse of the previous result:

Theorem (Necessary Condition [BGR14])

Let T be a stably infinite, universal, convex theory admitting quantifier-free interpolation and let Σ be a signature disjoint from the signature of T containing at least a unary predicate symbol. Then, $T \cup \mathcal{EUF}(\Sigma)$ has ***quantifier-free interpolation iff*** *T is* ***equality interpolating****.*

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$$\forall \underline{y}\, \forall \underline{y}' (\phi(\underline{x}, \underline{y}) \land \phi(\underline{x}, \underline{y}') \to y_i = y_i')$$

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The following Lemma supplies terms used as ingredients in the combined covers algorithm:

Lemma (Useful Terms)

Let $L_{i1}(\underline{x}) \vee \cdots \vee L_{ik_i}(\underline{x})$ *be the disjunctive normal form (DNF) of* $\texttt{ImplDef}^T_{\phi,y_i}(\underline{x})$. *Then, for every* $j = 1, \ldots, k_i$, *there is a* $\Sigma(\underline{x})$*-term* $t_{ij}(\underline{x})$ *such that* $T \vdash L_{ij}(\underline{x}) \wedge \phi(\underline{x}, \underline{y}) \rightarrow y_i = t_{ij}$

The terms t_{ij} are obtained thanks to the Beth definability property, that holds because of the Key Theorem.

Computing Combined Covers

- Given a Σ_1-theory T_1 and a Σ_2-theory T_2, we want to compute a $T_1 \cup T_2$-**cover** for $\exists \underline{e}\, \phi(\underline{x}, \underline{e})$ (Initial Formula).

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- The algorithm employs **acyclic explicit definitions** `ExplDef`$(\underline{z}, \underline{x})$ $\bigwedge_{i=1}^{m} z_i = t_i(z_1, \ldots, z_{i-1}, \underline{x})$ where the term t_i is pure.

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- A *working formula* is $\exists \underline{z}\, (\texttt{ExplDef}(\underline{z}, \underline{x}) \wedge \exists \underline{e}\, (\psi_1(\underline{x}, \underline{z}, \underline{e}) \wedge \psi_2(\underline{x}, \underline{z}, \underline{e})))$, where ψ_i is a Σ_i-formula $(i = 1, 2)$ and $\underline{x}$ are called *parameters*, $\underline{z}$ *defined variables* and $\underline{e}$ *(truly) existential variables*. ψ_1, ψ_2 always contain the literals $e_i \neq e_j$ (for distinct e_i, e_j from $\underline{e}$) as a conjunct.

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- A working formula is *terminal* iff for every $e_i \in \underline{e}$

$$T_1 \vdash \psi_1 \to \neg \texttt{ImplDef}^{T_1}_{\psi_1, e_i}(\underline{x}, \underline{z}) \text{ and } T_2 \vdash \psi_2 \to \neg \texttt{ImplDef}^{T_2}_{\psi_2, e_i}(\underline{x}, \underline{z})$$

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Start from an Initial Formula. The non-deterministic procedure to compute the terminal working formulae applies one of the following **alternatives**:

(1) Add to ψ_1 a disjunct from the DNF of $\bigwedge_{e_i \in \underline{e}} \neg \mathtt{ImplDef}^{T_1}_{\psi_1, e_i}(\underline{x}, \underline{z})$ and to ψ_2 a disjunct from the DNF of $\bigwedge_{e_i \in \underline{e}} \neg \mathtt{ImplDef}^{T_2}_{\psi_2, e_i}(\underline{x}, \underline{z})$;

(2.i) Select $e_i \in \underline{e}$ and $h \in \{1, 2\}$; then add to ψ_h a disjunct L_{ij} from the DNF of $\mathtt{ImplDef}^{T_h}_{\psi_h, e_i}(\underline{x}, \underline{z})$; add $e_i = t_{ij}$ (where t_{ij} is the term mentioned in **Useful Terms Lemma**) to $\mathtt{ExplDef}(\underline{z}, \underline{x})$; the variable e_i becomes *defined*.

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The output is the disjunction of all possible outcomes.

Transfer of covers

Proposition

*A **cover** of a terminal working formula can be obtained by unravelling the explicit definitions of the variables* $\underline{z}$ *from* $\exists \underline{z}\,(\mathtt{ExplDef}(\underline{z}, \underline{x}) \wedge \theta_1(\underline{x}, \underline{z}) \wedge \theta_2(\underline{x}, \underline{z}))$*, where* $\theta_1(\underline{x}, \underline{z})$ *is the* T_1*-**cover** of* $\exists \underline{e} \psi_1(\underline{x}, \underline{z}, \underline{e})$ *and* $\theta_2(\underline{x}, \underline{z})$ *is the* T_2*-**cover** of* $\exists \underline{e} \psi_2(\underline{x}, \underline{z}, \underline{e})$*.*

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From the Main Lemma, the previous Proposition and the 'Covers and QE' Theorem, we get:

Theorem

Let T_1, T_2 *be* ***convex, stably infinite, equality interpolating, universal theories over disjoint signatures admitting a model completion****. Then* $T_1 \cup T_2$ *admits a model completion too. Covers in* $T_1 \cup T_2$ *can be effectively* ***computed*** *as shown above.*

Combined Algorithm: an Example

Let T_1 be $\mathcal{EUF}(\Sigma)$ and T_2 be linear real arithmetic.

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Consider the formula:

$$\exists e_1 \cdots \exists e_4 \quad \begin{pmatrix} e_1 = f(x_1) \ \wedge \ e_2 = f(x_2) \wedge \\ \wedge \ f(e_3) = e_3 \ \wedge \ \ f(e_4) = x_1 \wedge \\ \wedge \ x_1 + e_1 \leq e_3 \ \wedge \ e_3 \leq x_2 + e_2 \ \wedge \ e_4 = x_2 + e_3 \end{pmatrix}$$

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Applying exhaustively Step (1) and Step (2.i), we get:

$$\begin{aligned} & [x_2 = 0 \ \wedge \ f(x_1) = x_1 \ \wedge \ x_1 \leq 0 \ \wedge \ x_1 \leq f(0)] \vee \\ & \vee \ [x_1 + f(x_1) < x_2 + f(x_2) \ \wedge \ x_2 \neq 0] \vee \\ & \vee \begin{bmatrix} x_2 \neq 0 \wedge x_1 + f(x_1) = x_2 + f(x_2) \ \wedge \ f(2x_2 + f(x_2)) = x_1 \wedge \\ \wedge f(x_1 + f(x_1)) = x_1 + f(x_1) \end{bmatrix} \end{aligned}$$

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The necessity can be easily deduced from the Necessity Theorem for **Equality Interpolating**.

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Non-transfer of Covers in the Non-convex case

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Consider the **cover transfer** for $T_1 \cup T_2$, where:

- $T_1 :=$ integer difference logic $\mathcal{IDL}$ (integer numbers with successor and predecessor, 0 and the strict order $<$): it is *not* convex, but it satisfies the equality interpolating condition for non-convex theories.
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The counterexample still applies when replacing integer difference logic with *linear integer arithmetics*.

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Conclusions

- Problem of **combined covers.**
- Sufficient and necessary conditions for transferring covers to **combinations** in the *convex* case.
- General method and algorithm for computing **combined covers** for *convex* theories, based on the use of Beth definability.
- Non-transfer of covers in the *non-convex* case, in general.

Future Work

- Investigate **cover transfer** for 'tame' theory combinations (*codomain* sorts are shared): already available in the ArXiv version;
- **Cover transfer** properties for non-disjoint signatures combinations.

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