Algebraic properties of enriched Priestley spaces

Dirk Hofmann (joint work with Pedro Nora)

July 8, 2021

CIDMA, Department of Mathematics, University of Aveiro, Portugal
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Duality,

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Algebraic properties of enriched Priestley spaces

Duality, Order, (Co)algebras, Topology, and Related topics

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Terminal coalgebra?

A question

For “the” Vietoris functor V, is the category CoAlg(V) of coalgebras for V complete (or has at least a terminal object)?

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Recall

- For a functor $F\colon \mathbf{C} \longrightarrow \mathbf{C}$, a coalgebra

$$\begin{array}{ccc} FX & \qquad & FY \\ c \uparrow & & \uparrow d \\ X & & Y \end{array}$$

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- Think of "the" Vietoris functor as a "topological powerset functor".

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- Think of “the” Vietoris functor as a “topological powerset functor”.

Example

The powerset functor $\mathrm{P}\colon \mathbf{Set} \longrightarrow \mathbf{Set}$ does not admit a terminal coalgebra.

"Das Orginal"

For a compact Hausdorff space X, the classic Vietoris space[a] VX consists of the set of all closed subsets of X

$$VX = \{K \subseteq X \mid K \text{ is closed}\}$$

equipped with the "hit-and-miss topology" generated by the subbasis of sets of the form (where $U \subseteq X$ is open)

$$U^{\Diamond} = \{A \in VX \mid A \cap U \neq \varnothing\}, \qquad U^{\Box} = \{A \in VX \mid A \cap U^{\complement} = \varnothing\}.$$

We obtain $\mathrm{V} \colon \mathbf{CompHaus} \longrightarrow \mathbf{CompHaus}$.

[a] Vietoris, Leopold (1922). "Bereiche zweiter Ordnung". In: *Monatshefte für Mathematik und Physik* **32**.(1), pp. 258–280.

Vietoris functors on topological spaces

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Remark

This definition can be generalised to other topological spaces … but does not always define a functor!!

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We consider here the following two variants on **Top**:

- lower Vietoris: closed subsets, but only "miss topology".

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We consider here the following two variants on **Top**:

- lower Vietoris: closed subsets, but only “miss topology”.
- compact Vietoris: compact subsets, “hit-and-miss topology”.

Vietoris functors more abstract (?)

Covariant presheafs

Consider, for a topological space X: $X \longmapsto \mathbf{2}^X$

- The exponential is taken in **PsTop**.

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- The convergence of 2^X can be split into a function $\mu\colon \mathrm{U}(2^X) \longrightarrow 2^X$ and the order relation $\subseteq$:

$$\mathfrak{p} \to A \iff \mu(\mathfrak{p}) \subseteq A.$$

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We dualise the order but keep μ ...

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We dualise the order but keep μ … and obtain the lower Vietoris space.

Restricting to (stably) compact spaces

The lower Vietoris functor restricts to $\mathsf{V}\colon \mathbf{StablyComp} \longrightarrow \mathbf{StablyComp}$

(those topological spaces X where the convergence splits “nicely” into a compact Hausdorff topology $\alpha\colon \mathsf{U}X \longrightarrow X$ and a partial order $\leq$ on X)

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Restricting to (stably) compact spaces

The lower Vietoris functor restricts to $\mathsf{V}\colon \mathbf{StablyComp} \longrightarrow \mathbf{StablyComp}$ and can be transferred along the adjunction

$$\mathbf{CompHaus} \underset{\text{forgetful}}{\overset{\text{discrete}}{\underset{\bot}{\rightleftarrows}}} \mathbf{PosComp} \sim \mathbf{StablyComp}$$

which leads to the classic Vietoris functor $\mathsf{V}\colon \mathbf{CompHaus} \longrightarrow \mathbf{CompHaus}$.

What is known (to us)?

Theorem

The compact Vietoris functor $\mathsf{V}\colon \mathbf{Haus} \longrightarrow \mathbf{Haus}$ *preserves codirected limits. Hence,* $\mathsf{CoAlg}(\mathsf{V})$ *is complete.*[a]

[a] Zenor, Phillip (1970). “On the completeness of the space of compact subsets”. In: *Proceedings of the American Mathematical Society* **26**.(1), pp. 190–192.

Some references

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Abramsky, Samson (2005). "A Cook's Tour of the Finitary Non-Well-Founded Sets". In: *We Will Show Them! Essays in Honour of Dov Gabbay*. Ed. by S. Artemov, H. Barringer, and A. A. Garcez. Vol. 1. London: College Publications, pp. 1–18.

Kupke, Clemens, Kurz, Alexander, and Venema, Yde (2004). "Stone coalgebras". In: *Theoretical Computer Science* **327**.(1-2), pp. 109–134.

Remark

For $\mathsf{V}\colon \mathbf{BooSp} \longrightarrow \mathbf{BooSp}$, the dual equivalence

$$\mathsf{CoAlg}(\mathsf{V}) \sim \mathbf{BAO}^{\mathrm{op}}$$

follows immediately from Halmos[a] duality:

$$\mathbf{BooSp}_{\mathbb{V}} \sim \mathbf{FinSup}^{\mathrm{op}}_{\mathrm{BA}}.$$

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- Coalgebra $X \to \mathsf{V}X \quad = \quad$ endomorphism in $\mathbf{BooSp}_{\mathbb{V}}$.
- Boolean algebra with operator $\quad = \quad$ endomorphism in $\mathbf{FinSup}_{\mathrm{BA}}$.

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Objective

Develop a similar duality theory for $\mathbf{StablyComp}_{\mathbb{V}}$ and beyond …

Passing to all (ordered) compact spaces?

Remark

Consider now:

$$\mathbf{CompHaus} \underset{\hom(-,\mathbf{2})}{\overset{C=\hom(-,\mathbf{2})}{\rightleftarrows}} \mathbf{DL}^{\mathrm{op}} \quad (\bot)$$

$$X \xrightarrow{\eta_X} \hom(CX,\mathbf{2}), \qquad x \longmapsto \mathrm{ev}_x$$

$$f = \mathrm{ev}_f \cdot \eta_X : X \longrightarrow \mathbf{2}.$$

Then: η_X is an isomorphism $\iff$ $(f : X \longrightarrow \mathbf{2})_f$ is point separating $\iff$ X is a Boolean space.

Passing to all (ordered) compact spaces?

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Consider now:

$$\mathbf{CompHaus} \underset{\hom(-,[0,1])}{\overset{C=\hom(-,[0,1])}{\rightleftarrows}} \mathbf{DL}^{\mathrm{op}} \quad (\bot)$$

$$X \xrightarrow{\eta_X} \hom(CX,[0,1]), \qquad x \longmapsto \mathrm{ev}_x$$

$$f : X \longrightarrow [0,1] \quad \text{factors as} \quad \mathrm{ev}_f \circ \eta_X, \quad \hom(CX,[0,1]) \xrightarrow{\mathrm{ev}_f} [0,1].$$

Then: η_X is an embedding $\iff$ $(f : X \longrightarrow [0,1])_f$ is point separating

$\Longleftarrow$ Urysohn Lemma.

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$$f : X \to [0,1], \quad \mathrm{ev}_f : \hom(CX,[0,1]) \to [0,1], \quad \mathrm{ev}_f \circ \eta_X = f.$$

Then: η_X is an embedding $\iff$ $(f : X \longrightarrow [0,1])_f$ is point separating

$\Longleftarrow$ Urysohn Lemma.

Theorem

For every compact Hausdorff space, η_X is an isomorphism if we consider above distributive lattices with constants from $[0,1]$.[a]

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[a] Banaschewski, Bernhard (1983). "On lattices of continuous functions". In: *Quaestiones Mathematicæ* **6**.(1-3), pp. 1–12.

Our thesis …

… is that the passage from the two-element space **2** to the compact Hausdorff space $[0, 1]$ one one side of the duality should be matched by a move from ordered structures to

order structures "in the logic of $[0, \infty]$ or $[0, 1]$".

HOFMANN, DIRK and NORA, PEDRO (2018). "Enriched Stone-type dualities". In: *Advances in Mathematics* **330**, pp. 307–360.

Keep the definition, change the logic

Our thesis …

… is that the passage from the two-element space $\mathbf{2}$ to the compact Hausdorff space $[0,1]$ one one side of the duality should be matched by a move from ordered structures to

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Theorem

The functor **CompHaus** $\xrightarrow{C=\hom(-,[0,1])}$ $([0,1]\text{-}\mathbf{DL})^{\mathrm{op}}$ *is fully faithful.*[a]

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Definition

A quantale $\mathcal{V} = (\mathcal{V}, \otimes, k)$ is a complete lattice $\mathcal{V}$ equipped with a commutative monoid structure $\otimes$, with identity k, so that, for each $u \in \mathcal{V}$,

$$u \otimes - \colon \mathcal{V} \longrightarrow \mathcal{V} \quad \text{has a right adjoint} \quad \hom(u, -) \colon \mathcal{V} \longrightarrow \mathcal{V}.$$

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Definition

1. A $\mathcal{V}$-category is a pair (X, a) consisting of a set X and a map $a : X \times X \longrightarrow \mathcal{V}$ satisfying

$$k \leq a(x, x) \qquad \text{and} \qquad a(x, y) \otimes a(y, z) \leq a(x, z).$$

Example: $\mathcal{V}$ with $\hom(-, -)$.

Quantale-enriched categories

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2. A $\mathcal{V}$-functor $f : (X, a) \longrightarrow (Y, b)$ between $\mathcal{V}$-categories is a map $f : X \longrightarrow Y$ such that

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3. $\mathcal{V}$-categories and $\mathcal{V}$-functors define the category $\mathcal{V}$-**Cat**.

Examples

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1. The two element chain $\mathbf{2} = \{0, 1\}$ with $\otimes = \&$. Then **2-Cat** $\sim$ **Ord**.
2. The extended real half line $\overleftarrow{[0, \infty]}$ ordered by the "greater or equal" relation $\geqslant$ and
   - the tensor product given by addition $+$, denoted by $\overleftarrow{[0, \infty]}_+$;

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3. The unit interval $[0, 1]$ with the "greater or equal" relation $\geqslant$ and the tensor $u \oplus v = \min\{1, u + v\}$, denoted as $\overleftarrow{[0, 1]}_\oplus$.

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4. The unit interval $[0, 1]$ with the usual order $\leqslant$ and $\otimes = \wedge$ the minumum, or $\otimes = *$ the usual multiplication, or $\otimes = \odot$ the Lukasiewicz sum defined by $u \odot v = \max\{0, u + v - 1\}$.

   Then $[0, 1]_\wedge$-**Cat** $\sim$ **UMet**, $[0, 1]_*$-**Cat** $\sim$ **Met**, $[0, 1]_\odot$-**Cat** $\sim$ **BMet**.

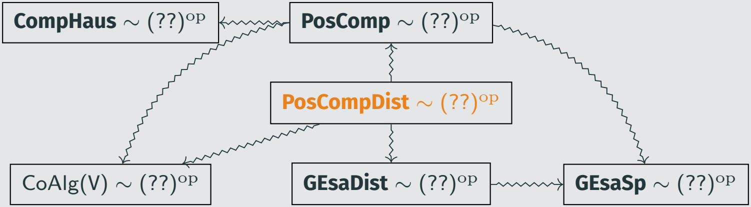
Remark

There are corresponding notions to most of classic notions from order theory:

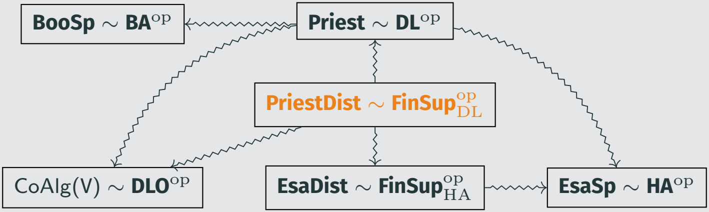
- (down/up)-closed subset: $X \longrightarrow \mathcal{V}$,
- supremum/infimum: weighted (co)limits,
- (co)completeness,
- Cauchy comleteness,
- (complete) distributivity,
- adjunction,
- ...

Our goal(s)

Metric Stone-type dualities

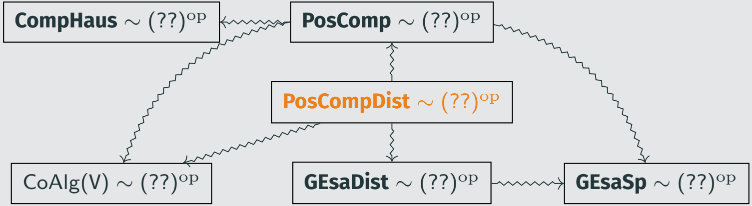


Here each "question mark category" should be a "$\mathcal{V}$-categorical" counterpart of the corresponding category in the "ordered picture".



Our goal(s)

Metric Stone-type dualities



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We should(?) also generalise the left-hand side:

- ordered compact space $\rightsquigarrow$ (certain) $\mathcal{V}$-categorical compact space

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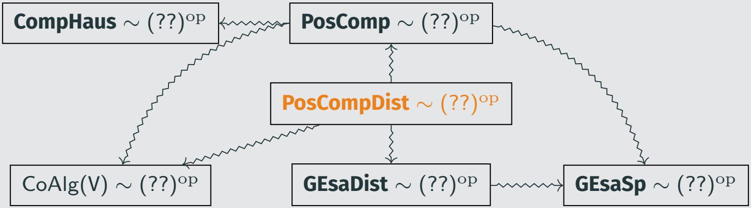

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  $X \times X \longrightarrow \mathbf{2}$ $\rightsquigarrow$ $X \times X \longrightarrow \mathcal{V}$
- Vietoris space $VX = \{X \to \mathbf{2}\}$ $\rightsquigarrow$ $VX = \{X \to \mathcal{V}\}$
- relation $X \times Y \longrightarrow \mathbf{2}$ $\rightsquigarrow$ $\mathcal{V}$-relation $X \times Y \longrightarrow \mathcal{V}$.

Recall

- Ordered topological structures are sets X equipped with order and topology so that the order relation is closed in $X \times X$.

Nachbin, Leopoldo (1950). *Topologia e Ordem.* University of Chicago Press.

Tholen, Walter (2009). "Ordered topological structures". In: *Topology and its Applications* **156**.(12), pp. 2148–2157.

Ordered compact Hausdorff spaces

Recall

- Ordered topological structures are sets X equipped with order and topology so that the order relation is closed in $X \times X$.
- For a compact Hausdorff topology $\alpha \colon \mathsf{U}X \longrightarrow X$ and an order relation $\leq \colon X \longrightarrow\!\!\!\!\!\!\mapsto X$, the following are equivalent.
  (i) The order $\leq$ is closed in $X \times X$.
  (ii) $\alpha \colon (\mathsf{U}X, \mathsf{U}{\leq}) \longrightarrow (X, \leq)$ is monotone.

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- An ordered compact Hausdorff space is Priestley if and only if the family $(f\colon X \longrightarrow \mathbf{2}^{\mathrm{op}})_f$ in **OrdCH** is point-separating and initial.

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Some conditions on $\mathcal{V}$

Assumption

From now on we work with a (ccd) quantale $\mathcal{V}$.

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Remark

- The Lawson topology on $\mathcal{V}$ is compact Hausdorff; with respect to this topology, an ultrafilter $\mathfrak{v}$ in $\mathcal{V}$ converges to

$$\xi(\mathfrak{v}) = \bigwedge_{A \in \mathfrak{v}} \bigvee A \in \mathcal{V}.$$

Note:

Scott topology:	$\mathfrak{v} \longrightarrow x \iff \xi(\mathfrak{v}) \geq x$
Dual of Scott topology:	$\mathfrak{v} \longrightarrow x \iff \xi(\mathfrak{v}) \leq x$

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form a subbase for the closed sets of the dual of the Scott topology of $\mathcal{V}$. We denote (the convergence of) this topology by $\xi_{\leq}$.

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- The convergence $\xi \colon \mathsf{U}\mathcal{V} \longrightarrow \mathcal{V}$ together with the ultrafilter monad $\mathbb{U} = (\mathsf{U}, m, e)$ and the quantale $\mathcal{V}$ defines a topological theory and therefore allows for an extension of the ultrafilter monad $\mathbb{U}$ to $\mathcal{V}$-**Cat**.

$\mathcal{V}$-categorical compact Hausdorff spaces

Definition

A $\mathcal{V}$-categorical compact Hausdorff spaces is a triple (X, a, α) where

- (X, a) is a $\mathcal{V}$-category and
- $\alpha\colon \mathsf{U}X \longrightarrow X$ is the convergence of a compact Hausdorff topology on X such that $\alpha\colon (\mathsf{U}X, \mathsf{U}a) \longrightarrow (X, a)$ is a $\mathcal{V}$-functor.

We denote the corresponding category as $\mathcal{V}$-**CatCH**.

Example

For $\mathcal{V} = \mathbf{2}$, we obtain Nachbin's ordered compact Hausdorff spaces.

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- These spaces should be thought of as natural generalisations of compact metric spaces.

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- For $\mathcal{V} = \overleftarrow{[0, \infty]}_+$, we obtain metric spaces equipped with a compatible compact Hausdorff topology.
- These spaces should be thought of as natural generalisations of compact metric spaces.
- For instance, the underlying metric of a metric compact Hausdorff space is Cauchy-complete, generalising the classic result that every compact metric space is Cauchy-complete.

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A $\mathcal{V}$-categorical compact Hausdorff space X is called Priestley whenever the cone $(f : X \longrightarrow \mathcal{V}^{\mathrm{op}})_f$ in $\mathcal{V}$-**CatCH** is point-separating and initial.

Proposition

For a $\mathcal{V}$-category (X, a) and a compact Hausdorff space (X, α) with the same underlying set X, the following assertions are equivalent.

(i) $\alpha\colon \mathsf{U}(X, a) \longrightarrow (X, a)$ *is a $\mathcal{V}$-functor.*

(ii) $a\colon (X, \alpha) \times (X, \alpha) \longrightarrow (\mathcal{V}, \xi_{\leq})$ *is continuous.*

Example: a duality result

Assumption

We consider only the Łukasiewicz tensor $\otimes = \odot$ on $[0,1]$, and take the enriched Vietoris monad where $\mathsf{V}X = \{X \to [0,1]\}$.

Theorem

The functor

$$([0,1]\text{-}\mathbf{Priest})_{\mathbb{V}} \xrightarrow{C=\hom(-,1)} [0,1]\text{-}\mathbf{FinSup}^{\mathrm{op}}$$

is fully faithful and restricts to a fully faithful functor

$$[0,1]\text{-}\mathbf{Priest} \xrightarrow{C=\hom(-,[0,1])} [0,1]\text{-}\mathbf{FinLat}^{\mathrm{op}}.$$

About (quasi)-varieties

Remark

The classic Stone duality

$$\mathbf{BooSp}^{\mathrm{op}} \sim \mathbf{BA}$$

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This can be also seen abstractly:

Theorem

A complete and cocomplete category is a finitary variety iff it has

1. *a finitely presentable, regularly projective regular generator, and*
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This can be also seen abstractly:

- **BooSp** has all colimits and limits,
- regular monomorphism $=$ subspace embedding,
- the two-element space is a regularly injective regular cogenerator,
- the two-element space is finitely copresentable.

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“Warm Up”: $\mathcal{V}$-categories

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*$\mathcal{V}$-**Cat**$^{\mathrm{op}}$ is a quasivariety.* *If time $\ll$ 18 mins then* skip

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- $\mathcal{V}_I$ (indiscrete structure) is a cogenerator.
- Hence $\mathcal{V} \times \mathcal{V}_I$ is a regular injective regular cogenerator.
- There is no rank since the “discrete” functor $\mathrm{D} \colon \mathbf{Set} \longrightarrow \mathcal{V}$-**Cat** preserves non-empty limits:

$$\hom(-, |X|) \simeq \hom(D-, X).$$

Therefore $|-| \colon \mathcal{V}$-**Cat** $\longrightarrow$ **Set** preserves copresentable objects. □

“Warm Up”: $\mathcal{V}$-categories

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$\mathcal{V}$-**Cat**$^{\mathrm{op}}$ *is a quasivariety.* *If time* $\ll$ 18 *mins then* skip

Theorem (just for comparision)

Top$^{\mathrm{op}}$ *is a quasivariety.*

Barr, Michael and Pedicchio, M. Cristina (1995). “Top$^{\mathrm{op}}$ is a quasi-variety”. In: *Cahiers de Topologie et Géométrie Différentielle Catégoriques* **36**.(1), pp. 3–10.

"Warm Up": $\mathcal{V}$-categories

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$\mathcal{V}$-$\mathbf{Cat}^{\mathrm{op}}$ is a quasivariety. *If time $\ll$ 18 mins then* skip

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$\mathbf{Top}_0^{\mathrm{op}}$ and $\mathcal{V}$-$\mathbf{Cat}_{\mathrm{sep}}^{\mathrm{op}}$ do not seem to be quasivarieties.

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However, **Pos**$^{\mathrm{op}}$ is a quasivariety. Are we missing something …??

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However, $\mathbf{Pos}^{\mathrm{op}}$ is a quasivariety. Are we missing something ...??

Theorem

The $\mathcal{V}$*-category* $\mathcal{V}$ *is a regular injective regular cogenerator in* $\mathcal{V}$-$\mathbf{Cat}_{\mathrm{sep,cc}}$*. Hence,* $(\mathcal{V}\text{-}\mathbf{Cat}_{\mathrm{sep,cc}})^{\mathrm{op}}$ *is a quasivariety.*

From Gabriel and Ulmer (1971)

Theorem

1. *A set X is copresentable in* **Set** *if and only if* $\mathrm{card}(X) = 1$.

Gabriel, Peter and Ulmer, Friedrich (1971). *Lokal präsentierbare Kategorien.* Lecture Notes in Mathematics, Vol. 221. Berlin: Springer-Verlag. v + 200.

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Theorem

1. *A set X is copresentable in* **Set** *if and only if* $\mathrm{card}(X) = 1$.
2. *The finitely copresentable compact Hausdorff spaces are precisely the finite ones.*
3. *The $\aleph_1$-copresentable compact Hausdorff spaces are precisely the metrisable ones. In particular, the unit interval $[0, 1]$ is $\aleph_1$-copresentable in* **CompHaus**.

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Theorem

The "connected component functor" $\pi_0 :$ **CompHaus** $\longrightarrow$ **BooSp** *preserves cofiltered limits.*

An immediate consequence for $\mathcal{V}$-CatCH

Remark

In

$$\mathbf{CompHaus} \underset{|-|}{\overset{\mathsf{D}}{\rightleftarrows}} \mathcal{V}\text{-}\mathbf{CatCH}_{(\mathrm{sep})} \quad (\mathsf{D} \dashv |-|),$$

the “discrete” functor $\mathsf{D}\colon \mathbf{CompHaus} \longrightarrow \mathcal{V}\text{-}\mathbf{CatCH}_{(\mathrm{sep})}$ preserves non-empty limits.

An immediate consequence for $\mathcal{V}$-CatCH

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the "discrete" functor $\mathsf{D}\colon \mathbf{CompHaus} \longrightarrow \mathcal{V}\text{-}\mathbf{CatCH}_{(\mathrm{sep})}$ preserves non-empty limits.

Corollary

For every regular cardinal λ, $|-|\colon \mathcal{V}\text{-}\mathbf{CatCH} \longrightarrow \mathbf{CompHaus}$ preserves λ-copresentable objects.

In particular:

1. *Every finitely copresentable $\mathcal{V}$-categorical compact Hausdorff space is finite*
2. *Every $\aleph_1$-copresentable $\mathcal{V}$-categorical compact Hausdorff space has a metrizable topology.*

Even more conditions on $\mathcal{V}$

Assumption

We also assume that there is a countable subset $D \subseteq \mathcal{V}$ so that, for all $v \in \mathcal{V}$,

$$v = \bigvee \{u \in D \mid u \lll v\}.$$

Proposition

A subbase for the Lawson topology on $\mathcal{V}$ is given by the sets

$$\{u \in \mathcal{V} \mid v \lll u\} \quad \text{and} \quad \{u \in \mathcal{V} \mid v \not\leq u\} \qquad (v \in D).$$

Hence, the Lawson topology on $\mathcal{V}$ has a countable base and therefore is metrisable.

Corollary

The compact Hausdorff space $\mathcal{V}$ (with the Lawson topology) is $\aleph_1$-copresentable in **CompHaus**.

Proposition

1. *$\mathcal{V}$-**Cat** is the model category of a countable $\aleph_1$-ary limit sketch in **Set**.*[a]

[a] Kelly, G. Max and Lack, Stephen (2001). "V-Cat is locally presentable or locally bounded if V is so". In: *Theory and Applications of Categories* **8**.(23), pp. 555–575.

Proof.

Use the bijection between the sets

$$\{X \to \mathcal{V}\} \qquad \text{and} \qquad \{(B_u)_{u \in D} \mid B_u \subseteq X \;\&\; B_u = \bigcap_{v \lll u} B_v\};$$

$$(\varphi\colon X \to \mathcal{V}_{\leq}) \longmapsto (\varphi^{-1}(\uparrow u)_{u \in D})$$

$$(B_u)_{u \in D} \longmapsto (\varphi\colon X \to \mathcal{V}, x \mapsto \bigvee\{u \in D \mid x \in B_u\})$$

then a map $a\colon X \times X \longrightarrow \mathcal{V}$ corresponds to a family $(R_u)_{u \in D}$ of binary relations R_u on X.

$\square$

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1. *$\mathcal{V}$-**Cat** is the model category of a countable $\aleph_1$-ary limit sketch in* **Set**.
2. *$\mathcal{V}$-**CatCH** is the model category of a countable $\aleph_1$-ary limit sketch in* **CompHaus**.

Proof.

Use the bijection between the sets

$$\{X \to \mathcal{V}_{\leq} \text{ continuous}\} \quad \text{and} \quad \{(B_u)_{u\in D} \mid B_u \subseteq X \text{ closed \& } B_u = \bigcap_{v \lll u} B_v\};$$

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$$(B_u)_{u\in D} \longmapsto (\varphi\colon X \to \mathcal{V}, x \mapsto \bigvee\{u \in D \mid x \in B_u\})$$

then a continuous map $a\colon (X,\alpha)\times(X,\alpha) \longrightarrow (\mathcal{V},\xi_{\leq})$ corresponds to a family $(R_u)_{u\in D}$ of closed binary relations R_u on X.

□

Proposition

1. $\mathcal{V}$-**Cat** *is the model category of a countable* $\aleph_1$*-ary limit sketch in* **Set**.
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Remark

Therefore $(\mathcal{V}\text{-}\mathbf{CatCH})^{\mathrm{op}}$ is the model category of a colimit sketch in the locally $\aleph_1$-presentable category $\mathbf{CompHaus}^{\mathrm{op}}$ and therefore locally presentable (we don't know the rank).[a]

[a] Adámek, Jiří and Rosický, Jiří (1994). *Locally presentable and accessible categories.* Vol. 189. London Mathematical Society Lecture Note Series. Cambridge: Cambridge University Press. xiv + 316, Remark 2.63.

Using limit Sketches

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Lemma

Let λ be a regular cardinal and let $\mathcal{S} = (\mathbf{C}, \mathcal{L}, \sigma)$ be a λ-small limit sketch. Then a model of $\mathcal{S}$ in a category $\mathbf{X}$ is λ-copresentable in $\mathrm{Mod}(\mathcal{S}, \mathbf{X})$ provided that each component is λ-copresentable in $\mathbf{X}$.

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Corollary

*An object is $\aleph_1$-ary copresentable in $\mathcal{V}$-**CatCH** if and only if its underlying compact Hausdorff space is metrizable.*

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*If the quantale $\mathcal{V}$ is finite, then the finitely copresentable objects of $\mathcal{V}$-**CatCH** are precisely the finite ones.*

Using limit Sketches

Proposition

1. *$\mathcal{V}$-**Cat** is the model category of a countable $\aleph_1$-ary limit sketch in **Set**.*
2. *$\mathcal{V}$-**CatCH** is the model category of a countable $\aleph_1$-ary limit sketch in **CompHaus**.*

Lemma

Let λ be a regular cardinal and let $\mathcal{S} = (\mathbf{C}, \mathcal{L}, \sigma)$ be a λ-small limit sketch. Then a model of $\mathcal{S}$ in a category $\mathbf{X}$ is λ-copresentable in $\mathrm{Mod}(\mathcal{S}, \mathbf{X})$ provided that each component is λ-copresentable in $\mathbf{X}$.

Remark

The conclusion of the lemma above is not necessarily optimal:

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- $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ is $\aleph_1$-copresentable in **CompHaus**.
- Hence, we conclude that $\mathbb{T}$ is $\aleph_1$-copresentable in **CompHausAb**.

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The conclusion of the lemma above is not necessarily optimal:

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- Hence, we conclude that $\mathbb{T}$ is $\aleph_1$-copresentable in **CompHausAb**.
- However, by the famous Pontryagin duality theorem, $\mathbb{T}$ is even finitely copresentable in **CompHausAb**.

The "Bourbaki-criterion"

Theorem

Let $D\colon I \longrightarrow \mathbf{CompHaus}$ *be a cofiltered diagram. Then a cone* $(p_i\colon L \longrightarrow D(i))_{i\in I}$ *for* D *is a limit cone if and only if*

1. $(p_i\colon L \longrightarrow D(i))_{i\in I}$ *is mono and,*
2. *for every* $i \in I$*:* $\bigcap_{j\to i} \operatorname{im} D(j \to i) = \operatorname{im} p_i$.

   That is, "the image of each p_i *is as large as possible".*

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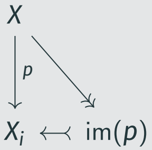
Example

Every Boolean space X is a cofiltered limit of finite spaces.

finite spaces:



Passing to Priestley spaces

Proposition

The reflection functor $\pi_0 \colon \mathcal{V}\text{-}\mathbf{CatCH} \longrightarrow \mathcal{V}\text{-}\mathbf{Priest}$ *preserves* $\aleph_1$*-cofiltered limits (and even cofiltered limits if* $\mathcal{V}$ *is finite).*

Proof.

… Gabriel and Ulmer (1971) use Stone duality … but here the "Bourbaki-criterion" also works … □

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Remark

For $\mathcal{V} = [0,1]$ and $\otimes = \odot$, we can use "Stone-duality": the dualising object $[0,1]$ induces a natural dual adjunction

$$[0,1]_\odot\text{-}\mathbf{CatCH} \underset{\hom(-,[0,1])}{\overset{C=\hom(-,[0,1])}{\rightleftarrows}} [0,1]_\odot\text{-}\mathbf{FinLat}^{\mathrm{op}} \quad (\perp)$$

where the fixed subcategory on the left-hand side is precisely $[0,1]_\odot$-**Priest**.

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1. *An object is* $\aleph_1$*-ary copresentable in* $\mathcal{V}$-**Priest** *if and only if its underlying compact Hausdorff space is metrizable. In particular,* $\mathcal{V}^{\mathrm{op}}$ *is* $\aleph_1$*-ary copresentable in* $\mathcal{V}$-**Priest**.

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2. *Assume that* $\mathcal{V}$ *is finite. Then an object is finitely copresentable in* $\mathcal{V}$-**Priest** *if and only if it is finite. In particular,* $\mathcal{V}^{\mathrm{op}}$ *is finitely copresentable in* $\mathcal{V}$-**Priest**.

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Theorem

The category $\mathcal{V}$-**Priest** *is locally* $\aleph_1$*-ary copresentable (and even locally finite copresentable if* $\mathcal{V}$ *is finite).*

Corollary

The fully faithful right adjoint functor

$$\mathsf{C}\colon [0,1]_{\odot}\text{-}\mathbf{Priest}^{\mathrm{op}} \longrightarrow [0,1]_{\odot}\text{-}\mathbf{FinLat}$$

preserves $\aleph_1$*-filtered colimits.*

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Corollary

The category $\mathsf{CoAlg}(\mathsf{V})$ *of coalgebras and homomorphisms for the enriched Vietoris functor* $\mathsf{V}\colon [0,1]_{\odot}\text{-}\mathbf{Priest} \longrightarrow [0,1]_{\odot}\text{-}\mathbf{Priest}$ *is locally* $\aleph_1$*-ary copresentable. In particular,* $\mathsf{CoAlg}(\mathsf{V})$ *is complete.*

Proof.

- Write **A** for the isomorphism closure of the image of

$$\mathsf{C}\colon ([0,1]_{\odot}\text{-}\mathbf{Priest})^{\mathrm{op}} \longrightarrow [0,1]_{\odot}\text{-}\mathbf{FinLat};$$

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- The category $\mathsf{Un}([0,1]_{\odot}\text{-}\mathbf{FinSup})$ of unary algebras and homomorphisms in $[0,1]_{\odot}$-**FinSup** is locally $\aleph_1$-ary presentable

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- The category $\mathsf{Un}([0,1]_{\odot}\text{-}\mathbf{FinSup})$ of unary algebras and homomorphisms in $[0,1]_{\odot}\text{-}\mathbf{FinSup}$ is locally $\aleph_1$-ary presentable and the forgetful functor $\mathsf{Un}([0,1]_{\odot}\text{-}\mathbf{FinSup}) \longrightarrow [0,1]_{\odot}\text{-}\mathbf{FinSup}$ preserves $\aleph_1$-filtered colimites.

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- $\mathsf{CoAlg}(V)^{\mathrm{op}}$ is equivalent to the category **B** obtained as the pullback

$$\begin{array}{ccc} \mathsf{Un}([0,1]_{\odot}\text{-}\mathbf{FinSup}) & \longleftarrow & \mathbf{B} \\ \downarrow & & \downarrow \\ [0,1]_{\odot}\text{-}\mathbf{FinSup} & \longleftarrow & \mathbf{A} \end{array}$$

of $\aleph_1$-accessible functors. □