A point-free approach to canonical extensions of Boolean algebras and lattice ordered-algebras

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DOCToR Workshop
7-9 July 2021

Canonical extensions of boolean algebras

We denote by BA the category of boolean algebras and boolean homomorphisms.

Definition Let $A, B \in \mathrm{BA}$ with B complete and $e : A \to B$ a boolean monomorphism.

- e is *dense* if each $b \in B$ is a join of meets from $e[A]$.
- e is *compact* if whenever $S, T \subseteq A$ with $\bigwedge e[S] \leq \bigvee e[T]$, then there are finite $S_0 \subseteq S$ and $T_0 \subseteq T$ with $\bigwedge S_0 \leq \bigvee T_0$.
- (B, e) is a canonical extension of A if e is dense and compact.

Example If X is the Cantor space and $B = \{\text{clopen subsets of } X\}$, then the inclusion $B \to \mathcal{P}(X)$ is a canonical extension.

If B is a boolean algebra and X_B its Stone space of ultrafilters, then the Stone map $\phi : B \to \mathcal{P}(X_B)$ sending b to $\phi(b) = \{x \in X_B : b \in x\}$ is a canonical extension of B.

Canonical extensions are unique up to isomorphism. The result above by Jónsson and Tarski shows under the Axiom of Choice (AC) that they exist.

Can their existence be seen choice-free?

Choice-free description of canonical extensions

N. Bezhanishvili and Holliday give the following description.

Let $B \in BA$ and X the set of proper filters of B. Viewing $(X, \subseteq)$ as a poset, topologize X with the Alexandroff topology of opsets of X.

Recall for a topological space Y, an open set is regular open if $U = \mathrm{Int}(\mathrm{cl}(U))$. The set $RO(Y)$ of regular open subsets of Y is a complete boolean algebra via

$$\bigvee \mathcal{U} = \mathrm{Int}(\mathrm{cl}(\bigcup \mathcal{U})) \qquad \bigwedge \mathcal{U} = \mathrm{Int}(\bigcap \mathcal{U}) \qquad \neg U = \mathrm{Int}(Y \setminus U)$$

Define $e : B \to RO(X)$ by $e(b) = \{ U \in RO(X) : b \in U \}$. BH show that e is a well-defined boolean monomorphism and that $(RO(X), e)$ is a canonical extension of B.

Their arguments are choice-free.

Can the description be made point-free by not referring to a topological space?

Point-free description of canonical extensions

We interpret $RO(X)$ purely frame-theoretically. Recall that a frame is a complete lattice L satisfying the infinite distributive law

$$a \wedge \bigvee S = \bigvee \{ a \wedge s : s \in S \} \text{ for all } a \in L \text{ and } S \subseteq L$$

Examples The collection of open subsets of a topological space with usual $\cup, \cap$ is a frame. In addition, complete boolean algebras are frames.

If L is a frame and $a \in L$, set $a^* = \bigvee \{ x \in L : a \wedge x = 0 \}$.

The set $B(L) = \{ a \in L : a^{**} = a \}$ is called the booleanization of L. It is a complete boolean algebra via

$$\bigsqcup S = (\bigvee S)^{**} \qquad \bigsqcap S = \bigwedge S \qquad \neg a = a^*$$

Definition For $A \in BA$ let $\mathrm{Filt}(A)$ be the set of filters of A, ordered by reverse inclusion.

$\mathrm{Filt}(A)$ is a bounded meet semilattice (SLat).

It is known that for a SLat M with top, the frame $Dn(M)$ of all downsets of M is the free frame on M, in that the forgetful functor from frames to SLat' has a left adjoint which sends M to $Dn(M)$.

Definition Let SLat'_0 be the category of bounded meet semilattices with meet semilattice homomorphisms preserving 0 and 1.

Theorem Let $A \in \mathrm{BA}$. The frame of upsets of proper filters of A, ordered by reverse inclusion, is the free frame on $\mathrm{Filt}(A) \in \mathrm{SLat}'_0$.

If $M \in \mathrm{SLat}'_0$ and F is the free frame on M, we denote by $\iota : M \to F$ the canonical SLat'_0- monomorphism.

Corollary If $A \in \mathrm{BA}$ and F is the free frame on $\mathrm{Filt}(A) \in \mathrm{SLat}'_0$, then the booleanization $B(F)$, together with the map $e : B \to B(F)$ defined by $e(b) = \iota(\uparrow b)$ is the canonical extension of B.

This description is then both choice-free and point-free.

We next address the analogous situation in the category bal of bounded archimedean lattice-ordered algebras.

We start by motivating and then defining this category.

Rings of continuous functions

If X is a topological space, set

$$C(X) = \{\text{real-valued continuous functions on } X\}$$

and

$$C^*(X) = \{\text{bounded elements of } C(X)\}.$$

These are commutative unital rings under pointwise $+, \cdot$ and are also $\mathbb{R}$-vector spaces.

$C(X)$ has a partial order $\leq$ given by $f \leq g$ if $f(x) \leq g(x)$ for all $x \in X$.

These are lattices via

$$(f \vee g)(x) = \max\{f(x), g(x)\} \qquad (f \wedge g)(x) = \min\{f(x), g(x)\}.$$

They are lattice-ordered algebras (ℓ-algebras for short), meaning they satisfy

- $f \leq g$ implies $f + h \leq g + h$
- $f \leq g$ and $0 \leq h$ implies $fh \leq gh$
- $f \leq g$ and $0 \leq r \in \mathbb{R}$ implies $rf \leq rg$.

Both $C(X)$ and $C^*(X)$ are archimedean, meaning that $nf \le g$ for all $n \ge 1$ implies that $f \le 0$.

$C^*(X)$ is bounded, meaning for each $f \in C^*(X)$ there is $n \ge 1$ with $|f| \le n$. Note that $|f| = f \vee (-f)$.

$C^*(X)$ has a norm given by $\|f\| = \sup\{|f(x)| : x \in X\}$. Alternatively, $\|f\| = \inf\{r \in \mathbb{R} : |f| \le r\}$.

Definition bal is the category of bounded archimedean ℓ-algebras with ℓ-algebra homomorphisms.

If $A \in$ bal, defining $\|a\| = \inf\{r \in \mathbb{R} : |a| \le r\}$ yields a well-defined norm.

Definition ubal is the full subcategory of bal consisting of those A which are complete with respect to the norm.

Gelfand Duality

There is a contravariant functor $C : \mathbf{KHaus} \to \mathbf{bal}$, where KHaus is the category of compact Hausdorff spaces and continuous maps. C sends $X \in \mathbf{KHaus}$ to $C(X)$ and a continuous map $\sigma : X \to Y$ to $C(\sigma) : C(Y) \to C(X)$, given by $C(\sigma)(f) = f \circ \sigma$.

Let $A \in \mathbf{bal}$. A ring ideal I of A is an ℓ-ideal if $|a| \leq |b|$ and $b \in I$ implies that $a \in I$.

Let $Y_A = \{$ maximal ℓ-ideals of $A \}$. If $M \in Y_A$, then $A/M \cong \mathbb{R}$.

With the Zariski topology, where closed sets have the form $Z(I) = \{ M \in Y_A : I \subseteq M \}$ for an ℓ-ideal I, $Y_A \in \mathbf{KHaus}$. It is called the Yosida space of A.

There is a contravariant functor $Y : \mathbf{bal} \to \mathbf{KHaus}$ sending A to Y_A and a bal-morphism $\alpha : A \to B$ to $Y(\alpha) : Y_B \to Y_A$, given by $M \mapsto \alpha^{-1}(M)$.

Gelfand Duality The functors C and Y form a contravariant adjunction between KHaus and bal which restricts to a dual adjunction between KHaus and ubal.

The natural isomorphism $J : 1_{\mathbf{bal}} \to CY$ of the adjunction is given by $J_A : A \to C(Y_A)$, $J_A(a)(M) = r$ if $a - r \in M$.

Dedekind and canonical extensions

Definition $D \in \mathbf{bal}$ is a Dedekind algebra if it is conditionally complete. That is, if each subset bounded above has a supremum in D.

Definition A Dedekind algebra D is the Dedekind completion of $A \in \mathbf{bal}$ if there is a bal-monomorphism $\alpha: A \to D$ such that each $d \in D$ is a join from $\alpha[A]$.

Existence of Dedekind completions follow from work of Nakano and Johnson. As a lattice they can be described as the set of normal lattice ideals of A.

Example Let $\alpha: A \to B$ be a bal-monomorphism with $A \neq 0$. Set $T = \{0\}$ and $S = \{r : 0 < r \in \mathbb{R}\}$. Then $\bigwedge \alpha[S] = 0 = \bigvee \alpha[T]$ but there is no finite $S_0 \subseteq S$ with $\bigwedge S_0 \leq \bigvee T$.

Definition Let $e: A \to B$ be a bal-monomorphism with B Dedekind.

- e is dense if each $b \in B$ is a join of meets from $\alpha[A]$.
- e is compact if for each $S, T \subseteq A$ and $0 < \varepsilon \in \mathbb{R}$ with $\bigwedge \alpha[S] + \varepsilon \leq \bigvee \alpha[T]$ there are finite $S_0 \subseteq S$ and $T_0 \subseteq T$ with $\bigwedge S_0 \leq \bigvee T_0$.
- (B, e) is a canonical extension of A if e is dense and compact.

If X is a set, then $B(X) = \{$ bounded real-valued functions $\}$ is a Dedekind algebra under pointwise operations.

Theorem (BMO) Let $A \in$ bal. Then $(B(Y_A), \mathfrak{J}_A)$ is the unique up to isomorphism canonical extension of A.

We wish to give a choice-free and a point-free description of the canonical extension of $A \in$ bal.

Definition An ℓ-ideal I is an *archimedean ideal* if A/I is archimedean.

Thus, I archimedean $\Rightarrow A/I \in$ bal. Given (AC) archimedean ideals are exactly intersections of maximal ℓ-ideals. This implies that there is a bijection between the set $\mathrm{Arch}(A)$ of archimedean ideals and the open sets of Y_A.

Let $X = \{$ proper archimedean ideals of $A \}$. We give X the Alexandroff topology. Existence of archimedean ideals does not require (AC). We use ideas of Dilworth to give a choice-free description of canonical extensions that is analogous to the BH description for boolean algebras.

Normal lower semicontinuous functions

Let Y be a topological space and f a bounded function on Y. For $x \in Y$ let $\mathcal{N}_x$ be the collection of open neighborhoods of x.

Definition $f^*(x) = \inf\{\sup f[U] : U \in \mathcal{N}_x\}$
and
$$f_*(x) = \sup\{\inf f[U] : U \in \mathcal{N}_x\}.$$

Then $f_* \leq f \leq f^*$, f_*, f^* are bounded, f^* is upper semicontinuous and f_* is lower semicontinuous. Also, f is lower semicontinuous iff $f = f_*$.

Definition $N(Y) = \{f \in B(Y) : f = (f^*)_*\}$. These are called normal functions.

Dilworth showed that if Y is completely regular, then $N(Y)$ is the Dedekind completion of $C^*(Y)$ as a lattice.

Dăneţ showed that there are operations on $N(Y)$ that make $N(Y)$ into a Dedekind algebra. These are the "normalizations" of the pointwise operations (e.g. $f \oplus g = ((f+g)^*)_*$).

Theorem For $A \in \mathbf{bal}$ and X the Alexandroff space of proper archimedean ideals of A, then $e: A \to N(X)$ is the canonical extension of A, where

$$e(a)(I) = \sup\{r \in \mathbb{R} : (r-a) \vee 0 \in I\} \quad \forall\ I \in X$$

This result is choice-free but not point-free. It is nontrivial to prove, including showing e is well-defined.

We wish to give a description of canonical extension that is both choice-free and point-free.

Theorem (Banaschewski) $\mathrm{Arch}(A)$ is a (compact, regular) frame.

Definition Let $\iota : \mathrm{Arch}(A) \to B$ be the free Boolean extension of $\mathrm{Arch}(A) \in \mathrm{DL}$.

That is, the forgetful functor $\mathrm{BA} \to \mathrm{DL}$ has a left adjoint, and this functor sends $\mathrm{Arch}(A)$ to B.

We also need the concept of a Specker algebra, whose description was first found by Bergman and independently by Rota.

Definition For $A \in \mathrm{BA}$ let $\mathbb{R}[A] = \mathbb{R}[\{ y_a : a \in A\}] / I$, the polynomial ring in variables $\{y_a : a \in A\}$, modulo the ideal generated by

$$y_{a \vee b} - (y_a + y_b - y_a y_b), \quad y_{a \wedge b} - y_a y_b, \quad y_{\neg a} - (1 - y_a), \quad y_0 \qquad (a, b \in A).$$

Theorem $\mathbb{R}[A] \in \mathrm{ba}\ell$ and $\mathrm{Id}(\mathbb{R}[A]) = \{ x_a : a \in A\} \cong A$, where $x_a = y_a + I$.

More general notions of Specker algebras where the base field could be any commutative ring, were studied in a paper by G. Bezhanishvili, Marra, –, and Olberding.

Theorem If B is the free boolean extension of $\mathrm{Arch}(A)$ and $D(\mathbb{R}[B])$ is the Dedekind completion of $\mathbb{R}[B]$, then $e: A \to D(\mathbb{R}[B])$ is the canonical extension of A, where

$$e(a) = -s + \bigvee \{ r\,\chi_{\neg\downarrow(I)} : I \in \mathrm{Arch}(A),\ (r-s-a)\vee 0 \in I \}.$$

and where $s \in \mathbb{R}$ satisfies $a+s \geq 0$.

The proof is very technical. Showing the join exists is not too bad. Showing the rest is all nontrivial.

A preprint is available on arXiv.org (with the same title as this talk).

Congratulations to Marco and Luca for completing their PhD degrees and for receiving postdocs in Salerno.