On Equational Completeness Theorems

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Equational completeness theorems

- The completeness theorem of intuitionistic propositional logic **IPC** w.r.t. the variety HA of Heyting algebras states that

$$\Gamma \vdash_{\mathbf{IPC}} \varphi \iff \text{for every } \boldsymbol{A} \in \mathsf{HA} \text{ and every hom } f \colon \boldsymbol{Fm} \to \boldsymbol{A}, \text{ if } f(\gamma) = 1 \text{ for all } \gamma \in \Gamma, \text{ then } f(\varphi) = 1.$$

- Given a class of similar algebras K and a set of equations $\Theta \cup \{\varphi \approx \psi\}$, we write $\Theta \vDash_{\mathsf{K}} \varphi \approx \psi$ when

$$\text{for every } \boldsymbol{A} \in \mathsf{K} \text{ and every hom } f \colon \boldsymbol{Fm} \to \boldsymbol{A}, \text{ if } f(\epsilon) = f(\delta) \text{ for all } \epsilon \approx \delta \in \Theta, \text{ then } f(\varphi) = f(\psi).$$

  When viewed as a relation, $\vDash_{\mathsf{K}}$ is called the

  equational consequence relative to K.

- In this terminology, the equational completeness theorem of **IPC** can be written, more concisely, as

$$\Gamma \vdash_{\mathbf{IPC}} \varphi \iff \{\gamma \approx 1 : \gamma \in \Gamma\} \vDash_{\mathsf{HA}} \varphi \approx 1.$$

- Furthermore, given a set of equations $\boldsymbol{\tau}(x)$ and a set of formulas $\Gamma \cup \{\varphi\}$, we write

$$\boldsymbol{\tau}(\varphi) := \{\epsilon(\varphi) \approx \delta(\varphi) : \epsilon \approx \delta \in \boldsymbol{\tau}\}$$
$$\boldsymbol{\tau}[\Gamma] := \bigcup_{\gamma \in \Gamma} \boldsymbol{\tau}(\gamma).$$

- Taking $\boldsymbol{\tau}(x) := \{x \approx 1\}$, we get

$$\Gamma \vdash_{\mathbf{IPC}} \varphi \iff \{\gamma \approx 1 : \gamma \in \Gamma\} \vDash_{\mathsf{HA}} \varphi \approx 1 \iff \boldsymbol{\tau}[\Gamma] \vDash_{\mathsf{HA}} \boldsymbol{\tau}(\varphi).$$

Rmk. The essence of this equational completeness theorem is that

IPC can be **interpreted** into $\vDash_{\mathsf{HA}}$.

This is made possible by translating formulas into equations by means of the set of equations $\boldsymbol{\tau}(x)$ as follows:

$$\psi \longmapsto \boldsymbol{\tau}(\psi), \text{ i.e., } \{\psi \approx 1\}.$$

- A (propositional) **logic** $\vdash$ is a consequence relation on the set of formulas of an arbitrary algebraic language that, moreover, is substitution invariant, i.e.,

$$\text{if } \Gamma \vdash \varphi, \text{ then } \sigma[\Gamma] \vdash \sigma(\varphi),$$

for every set of formulas $\Gamma \cup \{\varphi\}$ and every substitution σ.

Definition (Blok & Pigozzi)

A logic $\vdash$ admits an **equational completeness theorem** if there are a class of algebras K and a set of equations $\boldsymbol{\tau}(x)$ such that

$$\Gamma \vdash \varphi \iff \boldsymbol{\tau}[\Gamma] \vDash_{\mathsf{K}} \boldsymbol{\tau}(\varphi),$$

for every set of formulas $\Gamma \cup \{\varphi\}$.

Examples. **IPC** admits an equational completeness theorem w.r.t. the class of Heyting algebras. Similarly, every extension of **IPC** admits one w.r.t. an ISP-class of Heyting algebras.

Collateral damage.

- Glivenko's Theorem connects **CPC** and **IPC** as follows:

$$\Gamma \vdash_{\mathbf{CPC}} \varphi \iff \{\neg\neg\gamma : \gamma \in \Gamma\} \vdash_{\mathbf{IPC}} \neg\neg\varphi.$$

Thus, taking $\boldsymbol{\tau}(x) := \{\neg\neg x \approx 1\}$, we get

$$\begin{aligned}\Gamma \vdash_{\mathbf{CPC}} \varphi &\iff \{\neg\neg\gamma : \gamma \in \Gamma\} \vdash_{\mathbf{IPC}} \neg\neg\varphi \\ &\iff \{\neg\neg\gamma \approx 1 : \gamma \in \Gamma\} \vDash_{\mathsf{HA}} \neg\neg\varphi \approx 1 \\ &\iff \boldsymbol{\tau}[\Gamma] \vDash_{\mathsf{HA}} \boldsymbol{\tau}(\varphi),\end{aligned}$$

where HA is the variety of Heyting algebras.

Observation

CPC admits an equational completeness theorem w.r.t. the variety of **Heyting** algebras (although certainly not the intended one).

- Notably, the situation does not improve if we restrict to the case where $\boldsymbol{\tau}(x) = \{x \approx 1\}$. Actually, there is no escape from nonstandard equational completeness theorems.

Sometimes nonstandard equational completeness theorems are the sole possible ones. Let $\mathbf{CPC}_{\wedge\vee}$ be the $\langle \wedge, \vee \rangle$-fragment of **CPC**.

Observation

$\mathbf{CPC}_{\wedge\vee}$ does **not** admit any equational completeness theorem w.r.t. the variety of distributive lattices.

Proof.

- Suppose the contrary. Then there exists a set of equations $\boldsymbol{\tau}(x)$ witnessing an equational completeness theorem of $\mathbf{CPC}_{\wedge\vee}$ w.r.t. the variety DL of distributive lattices.
- As all equations in a single variable are valid in DL, we get

$$\mathsf{DL} \vDash \boldsymbol{\tau}(x), \text{ that is, } \emptyset \vDash_{\mathsf{DL}} \boldsymbol{\tau}(x).$$

- By the equational completeness theorem, $\emptyset \vDash_{\mathsf{DL}} \boldsymbol{\tau}(x)$ implies $\emptyset \vdash_{\mathbf{CPC}_{\wedge\vee}} x$, which is of course false. QED

Observation

$\mathbf{CPC}_{\wedge\vee}$ admits a (**nonstandard**) equational completeness theorem.

Proof sketch.

- Consider the three-element algebra

$$\boldsymbol{A} = \langle \{1, 0^+, 0^-\}; \wedge, \vee \rangle$$

with commutative operations defined by the tables

$\wedge$	0^-	0^+	1
0^-	0^+	0^+	0^+
0^+		0^-	0^+
1			1

$\vee$	0^-	0^+	1
0^-	0^+	0^+	1
0^+		0^-	1
1			1

- Then $\mathbf{CPC}_{\wedge\vee}$ admits an equational completeness theorem w.r.t. $\mathsf{K} := \{\boldsymbol{A}\}$ witnessed by the set of equations

$$\boldsymbol{\tau}(x) = \{x \approx x \wedge x\}.$$ QED

- Blok and Rebagliato generalized this construction to all logics with a lattice-based matrix semantics.

A modal example. The local consequence $\mathbf{K}_\ell$ of the modal system $\mathbf{K}$ is the logic defined as follows:

$$\Gamma \vdash_{\mathbf{K}_\ell} \varphi \iff \text{for all Kripke frame } \langle W, R\rangle, w \in W, \text{ and valuation } v, \text{ if } w, v \Vdash \Gamma, \text{ then } w, v \Vdash \varphi.$$

- One can replace $\mathbf{K}$ by $\mathbf{K4}$, $\mathbf{S4}$ (or any normal modal logic).
- By Blok and Rebagliato's trick, $\mathbf{K}_\ell$ admits an equational completeness theorem, but not a standard one:

Observation

The local consequence of the modal system $\mathbf{K}$ (resp. $\mathbf{K4}$, $\mathbf{S4}$) does **not** admit an equational completeness theorem w.r.t. the variety of modal algebras (resp. of K4-algebras, resp. of interior algebras).

Proof sketch.

- Suppose, by contradiction, that $\mathbf{K}_\ell$ admits an equational completeness theorem w.r.t. the variety of modal algebras MA.
- This must be witnessed by some $\boldsymbol{\tau}(x)$ containing an equation $\epsilon \approx \delta$ that fails in MA. Thus, w.l.o.g. $\epsilon \nvdash_{\mathbf{K}_\ell} \delta$.
- Since $\epsilon \approx \delta \in \boldsymbol{\tau}$, we get

$$x, \Box(\delta \to \delta) \vdash_{\mathbf{K}_\ell} \Box(\epsilon \to \delta).$$

- Then there are a Kripke frame $\langle W, R\rangle$, a valuation v, and a world $w \in W$ such that

$$w, v \Vdash \epsilon \text{ and } w, v \nVdash \delta.$$

- Attach to $\langle W, R\rangle$ a new point w^+ that sees everything.
- Extend the valuation v to the new frame, stipulating that x holds at w^+. Then

$$w^+, v \Vdash x \text{ and } w^+, v \nVdash \Box(\epsilon \to \delta).$$

- Thus, $x, \Box(\delta \to \delta) \nvdash_{\mathbf{K}_\ell} \Box(\epsilon \to \delta)$, a contradiction. QED

- On the other hand, some logics lack any equational completeness theorem.

Theorem (Raftery 2006)

Let $\vdash$ be a consistent logic whose language comprises only an implication connective $\to$. If $\vdash$ is weaker than or equal to the **relevance** logic **P–W** axiomatized by

$$\emptyset \rhd x \to x \qquad x, x \to y \rhd y,$$
$$\emptyset \rhd (x \to y) \to ((z \to x) \to (z \to y))$$
$$\emptyset \rhd (x \to y) \to ((y \to z) \to (x \to z)),$$

then $\vdash$ lacks any equational completeness theorem.

- Despite the simplicity of the concept, equational completeness theorems are poorly understood in general.

Aim of the talk

Characterize logics admitting an equational completeness theorem.

A general construction

Definition

Let $\vdash$ be a logic. Two formulas φ and ψ are **logically equivalent** if

$$\delta(\varphi, \vec{y}) \dashv\vdash \delta(\psi, \vec{y}),$$

for every formula $\delta(x, \vec{y})$. In this case, we write $\varphi \equiv_{\vdash} \psi$.

- In **IPC** or $\mathbf{K}_\ell$ this specializes to

$$\varphi \equiv \psi \Longleftrightarrow \emptyset \vdash \varphi \leftrightarrow \psi \Longleftrightarrow \varphi \dashv\vdash \psi.$$

Definition

A logic $\vdash$ is said to be **graph-based** if the arity of its connective is bounded above by one and, moreover, $\vdash$ has at most one unary connective.

Example. The $\langle \Diamond, 0, 1\rangle$-fragment of any modal logic is graph-based, while the $\langle \Diamond, \Box, 0, 1\rangle$-one is not.

Theorem

Let $\vdash$ a logic that is not graph-based. If there are two distinct logically equivalent formulas φ and ψ such that

$$\mathsf{Var}(\varphi) \cup \mathsf{Var}(\psi) = \{x\},$$

then $\vdash$ admits an equational completeness theorem.

Proof strategy.

- As $\vdash$ is not graph-based, it has either two distinct unary connectives $\Box$ and $\Diamond$, or an n-ary connective f with $n \geqslant 2$.
- Suppose the first case holds and let $k \in \mathbb{N}$ be greater than the length of all branches in the subformula trees of φ and ψ.
- Take

$$\varphi' := \Box^{2k}\Diamond\varphi(\Box^{k}\Diamond x) \text{ and } \psi' := \Box^{2k}\Diamond\psi(\Box^{k}\Diamond x).$$

and

$$\boldsymbol{\tau}(x) := \{\varphi' \approx \psi'\}.$$

- To prove that $\vdash$ admits an equational completeness theorem, it suffices to show that for all set of formulas $\Gamma \cup \{\gamma\}$,

$$\text{if } \boldsymbol{\tau}(\gamma) \subseteq \mathrm{Cg}^{\boldsymbol{Fm}}(\boldsymbol{\tau}[\Gamma]), \text{ then } \Gamma \vdash \gamma, \qquad (*)$$

  since, in this case, $\vdash$ has an equational completenss thm w.r.t.

$$\mathsf{K} := \{\boldsymbol{Fm}/\mathrm{Cg}^{\boldsymbol{Fm}}(\boldsymbol{\tau}[\Gamma]) : \Gamma \text{ is a theory of } \vdash\}$$

  witnessed by the set of equations $\boldsymbol{\tau}$.
- The hearth of the proof amounts to establishing $(*)$. But this is a problem about congruence generation. Then we use

Maltsev's Lemma

Let $\boldsymbol{A}$ be an algebra, $X \subseteq A \times A$, and $a, c \in A$. Then $\langle a, c\rangle \in \mathrm{Cg}^{\boldsymbol{A}}(X)$ if and only if there are $e_0, \dots, e_n \in A$, $\langle b_0, d_0\rangle, \dots, \langle b_{n-1}, d_{n-1}\rangle \in X$, and unary polynomial functions $p_0, \dots, p_{n-1}$ of $\boldsymbol{A}$ such that

$$a = e_0, c = e_n, \text{ and } \{e_i, e_{i+1}\} = \{p_i(b_i), p_i(d_i)\}, \text{ for every } i < n.$$

plus a combinatorial induction on subformula trees.

Corollary

Every logic is term-equivalent to one admitting an equational completeness theorem.

Proof.

- Take a logic $\vdash$ and add to its language two unary operations $\Box$ and $\Diamond$ that behave like the identity map.
- Clearly, $\vdash$ and the new logic $\vdash^+$ are term-equivalent.
- The new logic $\vdash^+$ is not graph-based (because its language comprises $\Box$ and $\Diamond$) and

$$\Box x \equiv_{\vdash} x \equiv_{\vdash} \Diamond x.$$

- Then $\vdash^+$ has two distinct logically equivalent formulas variable x, namely $\Box x$ and $\Diamond x$. By the previous Theorem, $\vdash^+$ admits an equational completeness theorem. QED

Observation. Admitting an equational completeness theorem is not a property of clones (no obvious decision procedure!).

Locally tabular logics

Definition

A logic $\vdash$ is **locally tabular** if for every $1 \leqslant n \in \mathbb{N}$ up to logical equivalence there are only finitely formulas in variables $x_1, \ldots, x_n$.

Example. Any logic complete w.r.t. a class of matrices whose algebraic reducts belong to a locally finite variety.

Corollary

Every locally tabular logic $\vdash$ that is not graph-based admits an equational completeness theorem.

Proof.

- Take a basic connective $f(x_1, \ldots, x_k)$ and define

$$\Box x := f(x, \ldots, x).$$

- By local tabularity, there are nonnegative integers $m \leqslant n$ s.t.

$$\Box^m x \equiv_{\vdash} \Box^{n+1} x.$$

- By the previous Theorem, we are done. QED

What about locally tabular graph-based logics? Take one such $\vdash$.

- The set of connectives of $\vdash$ comprises a single unary connective $\Box$ and some constants $\{\mathbf{c}_i : i < \alpha\}$.
- By local tabularity, there are $m \leqslant n$ such that $\Box^m x \equiv_{\vdash} \Box^{n+1} x$.

Theorem

$\vdash$ admits an equational completeness theorem iff there are $i < \alpha$ and $k \leqslant n$ s.t. either $x \vdash \Box x$ or $y \vdash \Box^k x$ or $y \vdash \Box^k \mathbf{c}_i$ or

1. $x, \Box^{t+k} x \dashv\vdash \Box^t \mathbf{c}_i, x$ for all $t \leqslant n$ and
2. for all $s, g, h, t \leqslant (2n - m + 1)^2$ and $\{u_j : s > j \in \omega\} \cup \{v_j : s > j \in \omega\} \subseteq \mathbb{N}$,

$$\{\Box^t x\} \cup \{\Box^{u_j} x : s > j \in \omega\} \cup \{\Box^{v_j} x : s > j \in \omega\} \vdash \Box^{t+g} x$$
$$\{\Box^{u_j} x : s > j \in \omega\} \cup \{\Box^{v_j} x : s > j \in \omega\} \vdash \Box^h \mathbf{c}_i,$$

provided that $u_j < v_j \leqslant n + n - m + 1$ for all $s > j \in \omega$, and that $\gcd(\{v_j - u_j : s > j \in \omega\})$ divides g and $h + k$.

- Thus, we have a complete description of locally tabular logics admitting an equational completeness theorem.
- Using the bounds for the graph-based case, one gets

Corollary

The problem of determining whether a locally tabular logic admits an equational completeness theorem is decidable.

Observation. In the above result logics can be presented either by a finite set of finite matrices or by a finite Hilbert calculus.

- The naive decision procedure runs in exponential time.

Open problem. Is this problem is complete for EXPTIME?

Logics with theorems

- A formula φ is a theorem of a logic $\vdash$ if $\emptyset \vdash \varphi$.

Definition

A logic $\vdash$ is **assertional** if there is a class of algebras K with a term-definable element $\top$ s.t. for all set of formulas $\Gamma \cup \{\varphi\}$,

$$\Gamma \vdash \varphi \iff \boldsymbol{\tau}[\Gamma] \vDash_{\mathsf{K}} \boldsymbol{\tau}(\varphi)$$

where $\boldsymbol{\tau}(x) := \{x \approx \top\}$.

- Assertional logics admit an equational completeness theorem by definition. They have theorems, since $\emptyset \vdash \top$.

Theorem (essentially Suszko)

A logic $\vdash$ is assertional iff it has theorems and

$$x, y, \delta(x, \vec{z}) \vdash \delta(y, \vec{z}),$$

for every formula $\delta(v, \vec{z})$.

- Logics with theorems admitting an equational completeness theorem can be described as follows:

Theorem

Let $\vdash$ a logic with a theorem ϵ such that $\mathrm{Var}(\epsilon) \neq \emptyset$. Then $\vdash$ admits an equational completeness theorem iff

1. $\vdash$ is graph-based and assertional; or
2. $\vdash$ is not graph-based and there are distinct logically equivalent formulas φ and ψ s.t.

$$\mathrm{Var}(\varphi) \cup \mathrm{Var}(\psi) = \{x\}.$$

Open problem. Extend this characterization beyond logics with theorems (ideally, to all logics).

Definition

A logic $\vdash$ is **protoalgebraic** if there is a set of formulas $\Delta(x, y)$ s.t.

$$\emptyset \vdash \Delta(x, x) \text{ and } x, \Delta(x, y) \vdash y.$$

Example. Almost every logic with a respectable implication connective $x \to y$. To see this, take

$$\Delta := \{x \to y\}.$$

Theorem

A nontrivial protoalgebraic logic $\vdash$ admits an equational completeness theorem iff there are two distinct logically equivalent formulas, that is, syntactic equality differs from logical equivalence.

- Essentially all reasonable protoalgebraic logics admit equational completeness theorems.
- However, **P–W** is a protoalgebraic logic lacking any equational completeness theorem.

Decision problems

We know that:

- Determining whether a logic admits an equational completeness theorem is decidable for locally tabular logics.

We shall see that:

- The same problem becomes undecidable for logics presented by an arbitrary Hilbert calculus.
- Undecidability persists if we restrict to protoalgebraic logics.

Strategy:

- We code the halting problem inside that of determining whether a protoalgebraic logic (presented by a finite Hilbert calculus) admits an equational completeness theorem.

Difficulty:

- A protoalgebraic logic admits an equational completeness theorem iff its has two distinct logically equivalent formulas. We need to code the halting problem without letting the logic know that word composition is associative.

Turing machines.

- A Turing machine $\mathbf{M}$ is a tuple $\langle P, Q, q_0, \delta \rangle$ where P and Q are sets of states, $q_0 \in Q$ is the initial state, Q the set of non-final states, P the set of final states, and

$$\delta \colon Q \times \{0, 1, \emptyset\} \to (Q \cup P) \times \{0, 1\} \times \{L, R\}.$$

- Instruction of the form $\delta(q, a) = \langle q', b, L \rangle$ mean: if the machine $\mathbf{M}$ reads a at state q, then it replaces a with b, moves left, and switches to state q'.
- Our Turing machines can write only zeros and ones, but can read 0, 1, and the empty symbol $\emptyset$.

Configurations.

- A configuration for $\mathbf{M}$ is a tuple $\langle q, \vec{w}, v, \vec{u} \rangle$ where $q \in Q \cup P$, $\vec{w}$ and $\vec{u}$ are either finite non-empty sequences of zeros and ones or the one-element sequence $\langle \emptyset \rangle$, and

$$v \in \{\langle 0 \rangle, \langle 1 \rangle, \langle \emptyset \rangle\}.$$

- Given two configurations $\mathtt{c}$ and $\mathtt{d}$ for $\mathbf{M}$, we say that $\mathtt{c}$ yields $\mathtt{d}$ if $\mathbf{M}$ allows to move from $\mathtt{c}$ to $\mathtt{d}$ in a single step.

- Let $\mathcal{L}(\mathbf{M})$ be the algebraic language with constant symbols in

$$P \cup Q \cup \{0, 1, \emptyset\},$$

a binary connective $x \cdot y$, and a ternary one $\lambda(x, y, z)$.

- The **logic of M** $\vdash_{\mathbf{M}}$ is axiomatized by the rules

$$q \cdot \lambda(x \cdot y, a, z) \rhd q' \cdot \lambda(x, y, b \cdot z)$$
$$\hat{q} \cdot \lambda(x, \hat{a}, y \cdot z) \rhd \hat{q}' \cdot \lambda(x \cdot \hat{b}, y, z)$$
$$p \cdot \lambda(x, y, z) \lhd \rhd p \cdot \lambda(\emptyset \cdot x, y, z)$$
$$p \cdot \lambda(x, y, z) \lhd \rhd p \cdot \lambda(x, y, z \cdot \emptyset)$$

for all $p, q, q', \hat{q}, \hat{q}' \in P \cup Q$ and $a, \hat{a}, b, \hat{b} \in \{0, 1, \emptyset\}$ s.t.

$$\delta(q, a) = \langle q', b, L \rangle \text{ and } \delta(\hat{q}, \hat{a}) = \langle \hat{q}', \hat{b}, R \rangle.$$

- We can code configurations $\mathtt{c} = \langle q, \vec{w}, v, \vec{u} \rangle$ for $\mathbf{M}$, where

$$\vec{w} = \langle w_1, \dots, w_n \rangle, v = \langle a \rangle, \text{ and } \vec{u} = \langle u_1, \dots, u_m \rangle,$$

with formulas of $\mathscr{L}(\mathbf{M})$ as follows:

$$\varphi_{\mathtt{c}} \coloneqq q \cdot \lambda((\cdots(w_1 \cdot w_2) \cdots) \cdot w_n, a, u_1 \cdot (\dots(u_{m-1} \cdot u_m) \dots)).$$

- If $\mathtt{c}$ yields $\mathtt{d}$, then $\varphi_{\mathtt{c}} \vdash_{\mathbf{M}} \varphi_{\mathtt{d}}$.

Let $\mathbf{M}$ be a Turing machine and $\vec{t}$ and input. Let also $\mathrm{In}(\mathbf{M}, \vec{t})$ be the initial configuration of $\mathbf{M}$ on input $\vec{t}$.

- Let $\vdash^{\vec{t}}_{\mathbf{M}}$ be the expansion of the logic of $\mathbf{M}$ with a new connective $\leftrightarrow$ and rules

$$\emptyset \rhd \varphi_{\mathrm{In}(\mathbf{M},\vec{t})}$$
$$p \cdot y \rhd x \leftrightarrow (x \cdot x)$$
$$\emptyset \rhd x \leftrightarrow x$$
$$x, x \leftrightarrow y \rhd y$$
$$x_1 \leftrightarrow y_1, \dots, x_n \leftrightarrow y_n \rhd *(x_1, \dots, x_n) \leftrightarrow *(y_1, \dots, y_n)$$

for every final state p, every positive integer n, and every n-ary connective $*$.

Notice that

- The last three rules guarantee that $\leftrightarrow$ captures logical equivalence, in the sense that

$$\epsilon \equiv_{\vdash^{\vec{t}}_{\mathbf{M}}} \delta \iff \emptyset \vdash^{\vec{t}}_{\mathbf{M}} \epsilon \leftrightarrow \delta.$$

- The logic $\vdash^{\vec{t}}_{\mathbf{M}}$ is protoalgebraic with $\Delta(x, y) := \{x \leftrightarrow y\}$.

Theorem

Let $\mathbf{M}$ a Turing machine and $\vec{t}$ an input. Then $\mathbf{M}$ halts on $\vec{t}$ iff the logic $\vdash^{\vec{t}}_{\mathbf{M}}$ admits an equational completeness theorem.

Proof sketch (easy part).

- If $\mathbf{M}$ halts on $\vec{t}$, there is a sequence of configurations $c_1, \dots, c_k$ s.t. $c_1 = \mathrm{In}(\mathbf{M}, \vec{t})$, each c_i yields c_{i+1}, and the state p of c_k is final.
- Since each c_i yields c_{i+1},

$$\varphi_{\mathrm{In}(\mathbf{M},\vec{t})} \vdash^{\vec{t}}_{\mathbf{M}} \varphi_{c_2} \vdash^{\vec{t}}_{\mathbf{M}} \cdots \vdash^{\vec{t}}_{\mathbf{M}} \varphi_{c_k}.$$

- By definition, $\emptyset \vdash^{\vec{t}}_{\mathbf{M}} \varphi_{\mathrm{In}(\mathbf{M},\vec{t})}$, whence $\emptyset \vdash^{\vec{t}}_{\mathbf{M}} \varphi_{c_k}$.
- Since $\varphi_{c_k} = p \cdot \psi$ for some ψ, we can apply the rule $p \cdot y \rhd x \leftrightarrow (x \cdot x)$, obtaining

$$\emptyset \vdash^{\vec{t}}_{\mathbf{M}} x \leftrightarrow (x \cdot x).$$

- Thus, x and $x \cdot x$ are logically equivalent distinct formulas. As $\vdash^{\vec{t}}_{\mathbf{M}}$ is protoalgebraic, it admits an equational completeness theorem. QED

Corollary

The problem of determining whether a logic presented by a finite Hilbert calculus admits an equational completeness theorem is **undecidable**.

Summary.
We characterized logics admitting an equational completeness theorem in the following settings:

- Locally tabular logics, logics with theorems, protoalgebraic logics.

The problem of determining whether a logic admits an equational completeness theorem is:

- Decidable for logics presented by a finite set of finite matrices and locally tabular logics presented by a finite Hilbert calculus;
- Undecidable for arbitrary (protoalgebraic) logics presented by a finite Hilbert calculus.

Open problems. Standard equational completeness theorem, complexity issues, logics lacking theorems etc.

Thank you very much for your attention!