A CATEGORICAL VIEW ON LOGICAL RESOURCES

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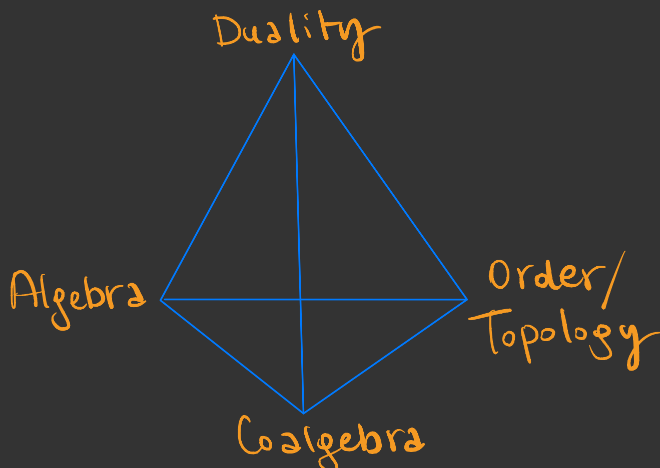
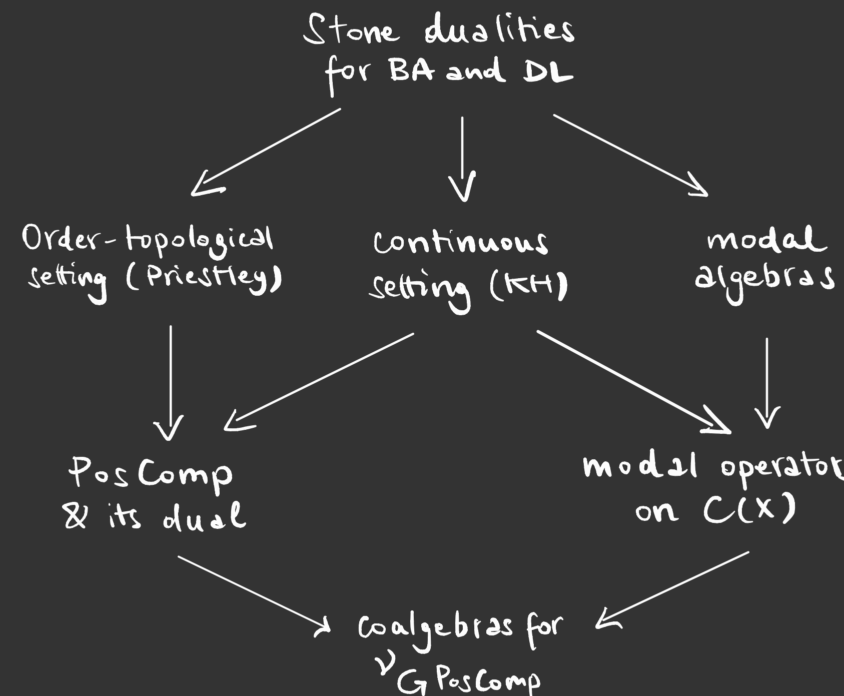
§1. STONE DUALITY AND BEYOND
(a slightly biased perspective)



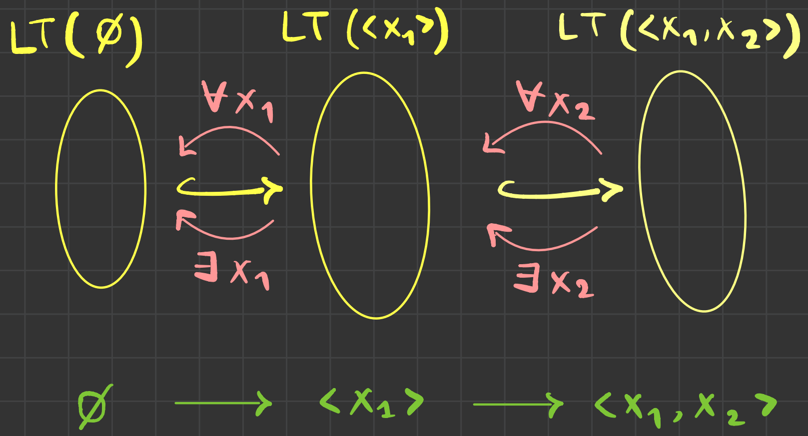


- (Infinitary) extensions of algebraic dualities — possibly with modal operators.

  Still relatively tractable, with useful applications, e.g. relating logical properties to algebraic and topological ones.

- The situation is remarkably different when we move from propositional to predicate logics

  - Cylindric algebras (Tarski, Monk) and polyadic BA (Halmos): quantification as extra structure
  - Lawvere's hyperdoctrines: quantification as a property
  - Makkai's ultracategories: one-sided duality for FO theories



A <u>(Boolean) hyperdoctrine</u> is a functor

$$\text{Contexts} \longrightarrow \text{BA}$$

satisfying the Beck-Chevalley condition ($\exists$ commutes with substitutions)

Applications of categorical semantics of predicate logics are few and far between. Notable exceptions include:

- Pitts, An application of open maps to categorical logic (JPAA 1983)
- Ghilardi & Zawadowski, Model completions and r-Heyting cat's (APAL 1997)
- Coumans, Generalising canonical extension to the categorical setting (APAL 2012)

- Lack of a usable duality for these categorical semantics

- Fundamental (open) problem: Characterise those categories which are (up to equivalence) of the form $\mathrm{Mod}(T)$ for some FO theory T.

  Partial results in this direction:

  - A category is locally finitely presentable iff it is equivalent to one of the form $\mathrm{Mod}(T)$ for T a limit theory.
    (A limit sentence is one of the form $\forall \bar{x}(\varphi(\bar{x}) \to \exists! \bar{y}\, \psi(\bar{x}, \bar{y}))$ where φ and ψ are conjunctions of atomic formulas)

  - For every FO theory T, the forgetful functor $\mathrm{Mod}(T) \longrightarrow \mathrm{Set}$ preserves directed colimits (this extends to AEC: Shelah, Rosicky, etc.)

  - Anti-elementary classes (Wehrung)

§2. LOGICAL RESOURCES

Hyperdoctrines typically stratify (algebras of) formulas in terms of contexts (i.e., tuples of free variables).

But there are other interesting and meaningful logical resources that one may want to consider, such as:

- quantifier-rank
- modal-depth in modal logic
- complexity of guards in guarded logics

This is especially relevant in finite model theory, where resource-bounded logics play an important role. However, it sometimes leads to improvements on — and finer analyses of — classical results in model theory (e.g., Rossman's Equirank HPT).

A categorical approach to the study of logical resources was recently introduced by Abramsky, Dawar et al.

It is based on the notion of game comonad.

- Abramsky, Dawar & Wang, The pebbling comonad in finite model theory (LICS 2017)
- Abramsky & Shah, Comonadic semantics for computational resources (CSL 2018)

The original intent was (and still is) to relate two (almost disjoint) strands in theoretical computer science, one focusing on semantics and compositionality (exemplified by categorical semantics), the other on complexity and expressiveness (exemplified by finite model theory).

I believe this novel approach may be of interest, more broadly, to those working in categorical, (co)algebraic and duality approaches to logic.

GAME COMONADS

$R(\sigma)$: category of relational structures for a finite relational signature σ and homomorphisms between them.

Suppose that we are interested in FO_k, first-order logic with quantifier-rank at most k. We then have limited access to a structure: the only properties that can be "seen" are those expressible in the fragment FO_k.

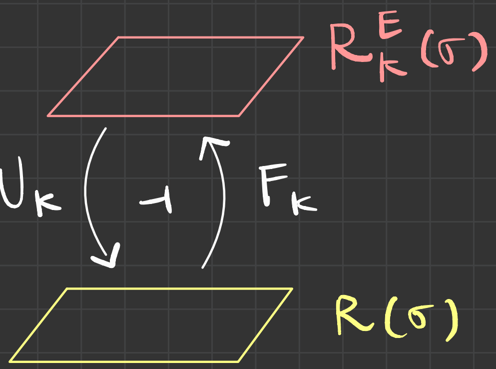
We consider a category of forest-ordered σ-structures $(A, \leq)$.

$R^E(\sigma)$:

- objects: pairs $(A, \leq)$ satisfying
  - (E): if $x, y \in A$ appear in a tuple of related elements, then either $x \leq y$ or $y \leq x$
- morphisms: homomorphism of σ-structures that are also forest morphisms

- The intuition is that the forest order on a σ-structure A describes the "accessible part" of A. In particular, we can restrict access by bounding the height of the forest orders. For all $k \geq 0$, set

$R_k^E(\sigma)$: full subcategory of $R^E(\sigma)$ on those $(A, \leq)$ s.t. the forest-order $\leq$ has height at most k.



The functor U_k, which forgets the forest order, has a right adjoint F_k which builds a forest-ordered structure in a (co-)free manner so as to satisfy (E).

Crucially, <u>U_k is comonadic</u>, i.e. $R_k^E(\sigma)$ is isomorphic to the category of coalgebras for the comonad $\mathbb{E}_k := U_k \circ F_k$.

$\mathbb{E}_k$ is the Ehrenfeucht-Fraïssé comonad on $R(\sigma)$.

$$\begin{tikzcd} R^E_k(\sigma) \arrow[d, bend right, "U_k"'] \arrow[d, phantom, "\dashv"] \\ R(\sigma) \arrow[u, bend right, "F_k"'] \end{tikzcd}$$

The following fall out of this picture:

- **LOGICAL EQUIVALENCES**

  We can define natural <u>resource-indexed equivalence relations</u> on $R(\sigma)$. E.g.,

  $\forall A, B \in R(\sigma)$, $A \rightleftarrows_k B$ iff $F_k(A) \rightleftarrows F_k(B)$.

> **Theorem (Abramsky, Shah)** The following hold for all $A, B \in R(\sigma)$:
>
> i) $A \rightleftarrows_k B$ iff $A \equiv^{\exists^+ FO_k} B$; ii) $A \leftrightarrow_k B$ iff $A \equiv^{FO_k} B$.
>
> Moreover, if A and B are finite, we have iii) $A \cong_k B$ iff $A \equiv^{FO_k(\#)} B$.

- **MODEL-COMPARISON GAMES**

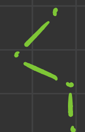
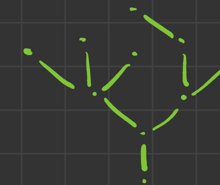
  Morphisms $\mathbb{E}_k A \to B$ (equivalently, morphisms $A \to B$ in the Kleisli category) can be seen as <u>winning strategies for Duplicator</u> in the one-sided Ehrenfeucht–Fraïssé game played between A and B.

- **COMBINATORIAL PARAMETERS**

  The coalgebra number of $A \in R(\sigma)$, if it exists, is the least $k \geq 0$ s.t. A admits a coalgebra structure for $\mathbb{E}_k$. For all finite $A \in R(\sigma)$, the coalgebra number of A exists and coincides with the <u>tree-depth</u> of A (Abramsky & Shah, 2018).

Similar results hold for other logic fragments, such as finite-variable logic and modal logic with bounded modal-depth. The latter are captured by the pebbling and modal comonads, respectively.

Moreover, many of these results can be derived axiomatically, working in arboreal categories (Abramsky & LR, 2021). These are based on an abstraction of the idea of forest-ordered covers of σ-structures.

Basic objects: Paths  Every object in an arboreal category is obtained as a canonical colimit of paths. 

Every arboreal category $\mathcal{C}$ induces a canonical functor $\mathcal{C} \longrightarrow \mathbf{Trees}$ which allows for notions such as back-and-forth systems and back-and-forth games to be defined in $\mathcal{C}$. Homomorphisms preservation theorems can then be formulated and proved in this axiomatic framework.

I think these ideas could be even more fruitful if combined with tools and intuitions from categorical logic and duality theory.

§3. HOMOMORPHISM COUNTING

As a first step towards the study of the relation between game comonads and other (already existing) tools of duality and categorical logic, we shall look at homomorphisms counting results.

"Unnatural Yoneda"

These can be of various kinds:

LOVÁSZ + PULTR

a) For any two finite graphs/groups/lattices/Boolean algebras... A and B,

$$A \cong B \iff |\mathrm{Hom}(A,C)| = |\mathrm{Hom}(B,C)| \quad \forall C \text{ finite graphs}/\dots$$
$$\iff |\mathrm{Hom}(C,A)| = |\mathrm{Hom}(C,B)| \quad \forall C \text{ finite graphs}/\dots$$

GROHE

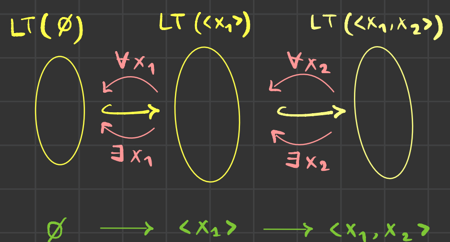
b) For any two finite σ-structures A and B,

$$A \equiv^{FO_k(\#)} B \iff |\mathrm{Hom}(C,A)| = |\mathrm{Hom}(C,B)| \quad \forall C \text{ finite with } td(C) \leq k$$

c) Two finitely generated profinite groups are (topologically) isomorphic iff they have the same finite (continuous, homomorphic) images.

Homomorphism counting results are important in finite model theory and graph theory because they allow for the study of structures using techniques from linear algebra. E.g. a finite graph G is determined, up to iso, by the vector $(|\mathrm{Hom}(G,H)|)_H \in \mathbb{N}^\omega \subseteq \mathbb{R}^\omega$, where H ranges over the set of (isomorphism classes of) finite graphs.

Naive idea: look at the functors $\mathrm{Hom}(-,A) : \mathscr{C}^{op} \longrightarrow \mathbf{FinSet}$ as some sort of "nice" presheaves (where, e.g., $\mathscr{C} = \mathrm{Grp}_{fin}$).



Taking the (pointwise) dual of Boolean hyperdoctrines, we obtain the notion of polyadic space (Joyal, 1971):

$$F : \mathscr{C}^{op} \longrightarrow \mathbf{Stone} \quad \text{satisfying} \begin{cases} \cdot \text{ AMALGAMATION} \\ \cdot \text{ OPENNESS} \end{cases}.$$

Restricting to the "finite duality", we have in particular polyadic finite sets:

$$F : \mathscr{C}^{op} \longrightarrow \mathbf{FinSet} \quad \text{satisfying AMALGAMATION}$$

Theorem [Lovász; Pultr; Isbell; Dawar, Jakl & LR; LR]

Let $\mathscr{C}$ be a locally finite category admitting a proper factorisation system $(\mathcal{Q}, \mathcal{M})$ s.t. $\mathscr{C}$ is $\mathcal{Q}$-well-founded. Then, for all $a, b \in \mathscr{C}$,

$$a \cong b \iff |\mathrm{Hom}(c,a)| = |\mathrm{Hom}(c,b)| \quad \forall c \in \mathscr{C}.$$

(For the dual version, replace "$\mathcal{Q}$-well-founded" with "$\mathcal{M}$-well-founded".)

Sketch of Proof: Let $F := \mathrm{Hom}(-,a)$ and $G := \mathrm{Hom}(-,b)$. Then $F, G \colon \mathscr{C}^{op} \to \mathbf{FinSet}$ are pointwise isomorphic polyadic finite sets.

Lemma: Two polyadic finite sets F, G are pointwise isomorphic iff so are their (Stirling) cores $F^{\bullet}, G^{\bullet}$. [$F^{\bullet}(c) \subseteq F(c)$ consists of the points of "maximal dimension"]

$\forall c \in \mathscr{C}$, $F(c) \cong G(c) \Rightarrow F^{\bullet}(c) \cong G^{\bullet}(c) \iff \mathcal{M}(c,a) \cong \mathcal{M}(c,b) \Rightarrow (c \rightarrowtail a \text{ iff } c \rightarrowtail b)$.

It follows that $a \rightleftarrows b$ and so $a \cong b$ (a finite monoid satisfying the (left) cancellation law is a group). $\square$

The previous homomorphism counting result can be extended to characterise "infinite objects that are determined by their finite approximations".

Typically, these are certain objects in $\mathrm{Ind}(\mathscr{C})$ or $\mathrm{Pro}(\mathscr{C})$, for $\mathscr{C}$ a locally finite category. We recover the results mention earlier (and obtain new ones):

a) For any two finite graphs/groups/lattices/Boolean algebras... A and B,

$\boxed{\text{LOVÁSZ + PULTR}}$

$$A \cong B \iff |\mathrm{Hom}(A,C)| = |\mathrm{Hom}(B,C)| \quad \forall C \text{ finite graphs/}\cdots$$
$$\iff |\mathrm{Hom}(C,A)| = |\mathrm{Hom}(C,B)| \quad \forall C \text{ finite graphs/}\cdots$$

b) For any two finite σ-structures A and B,

$\boxed{\text{GROHE}}$

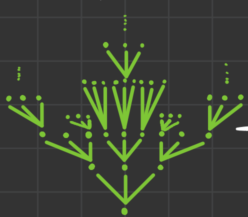
$$A \equiv^{FO_k(\#)} B \iff |\mathrm{Hom}(C,A)| = |\mathrm{Hom}(C,B)| \quad \forall C \text{ finite with } td(C) \le k$$

b): apply the theorem in the previous slide to the category of finite coalgebras for the Ehrenfeucht-Fraïssé comonad.

c) Two finitely generated profinite groups are (topologically) isomorphic iff they have the same finite (continuous, homomorphic) images.

c) Work in $\mathrm{Pro}(\mathrm{Grp}_{fin})$.

Also: - Finitely generated profinite groups/Abelian groups/$\mathbb{V}$-algebras are determined by hom-counts <u>to</u> finite (discrete) algebras.



- finitely branching trees are determined by hom-counts <u>from</u> finite trees.

CONCLUSIONS

- Better understanding of the relations between comonadic semantics for logical resources, categorical semantics à la Lawvere/Joyal, and duality?

- Logic-based proof of homomorphism counting results?

- Isomorphism vs elementary equivalence: these notions coincide for finite models, but also for finitely generated profinite groups (Jarden & Lubotzky). Can we understand this in terms of polyadic spaces?

Thank You!