Admissibility of Π_2-Inference Rules: interpolation, model completion, and contact algebras

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The use of non-standard rules has a long tradition in modal logic starting from the pioneering work of Gabbay [5], who introduced a non-standard rule for irreflexivity. While standard inference rules can be identified with universally quantified Horn formulas, non-standard rules correspond to formulas that allow extra universally quantified variables in their premises. Non-standard rules have been employed in temporal logic in the context of branching time logic [3] and for axiomatization problems [6] concerning the logic of the real line in the language with the Since and Until modalities. General completeness results for modal languages that are sufficiently expressive to define the so-called difference modality have been obtained in [13]. For the use of the non-standard density rule in many-valued logics we refer to [10] and [11].

Recently, there has been a renewed interest in non-standard rules in the context of the region-based theories of space [12]. One of the key algebraic structures in these theories is that of *contact algebras*. These algebras form a discriminator variety, see, e.g., [2]. Compingent algebras are contact algebras satisfying two $\forall\exists$-sentences (aka Π_2-sentences) [2, 4]. De Vries [4] established a duality between complete compingent algebras and compact Hausdorff spaces. This duality led to new logical calculi for compact Hausdorff spaces in [1] for a two-sorted modal language and in [2] for a uni-modal language with a strict implication. Key to these approaches is a development of logical calculi corresponding to contact algebras. In [2] such a calculus is called the *strict symmetric implication calculus* and is denoted by $\mathsf{S^2IC}$. The extra Π_2-axioms of compingent algebras then correspond to non-standard Π_2-rules, which turn out to be admissible in $\mathsf{S^2IC}$. This generates a natural question of investigating admissibility of Π_2-rules in $\mathsf{S^2IC}$ studied in [2] and in general in logical calculi corresponding to varieties of modal algebras. In fact, rather little is known about the problem of recognizing *admissibility* for such non-standard rules, although this problem has already been raised in [13]. This is the question that we address in this paper.

We undertake a systematic study of admissibility of Π_2-rules. We show that *there are tools already available in the literature on modal logic* that can be fruitfully employed for this aim: these tools include algorithms for deciding conservativity [7, 9], as well as algorithms for computing local and global interpolants. We devise three different strategies for recognizing admissibility of Π_2-rules over some system $\mathcal{S}$. The definition of Π_2-rules that we consider is taken from [2] and is close to that of Balbiani et al. [1].

The first strategy applies to a logic $\mathcal{S}$ with the interpolation property. We show that Π_2-rules are effectively recognizable in $\mathcal{S}$ in case $\mathcal{S}$ has the interpolation property and conservativity

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is decidable in $\mathcal{S}$. The second strategy applies to logics admitting local and global uniform interpolants, respectively. Global interpolants are strictly related to model completions and to axiomatizations of existentially closed structures [8], thus establishing a direct connection between Π_2-rules and model-theoretic machinery. Directly exploiting this connection leads to our third strategy. We apply the third strategy to our main case study to show admissibility of various Π_2-rules in $\mathsf{S^2IC}$, thus recovering admissibility results from [2] as special cases (we also show that the admissibility problem for $\mathsf{S^2IC}$ is co-NExpTime-complete). The model completion we use to this aim is that of the theory of contact algebras. Finally, we prove the technically most challenging result of our contribution: that the model completion of contact algebras is finitely axiomatizable. As a consequence of this result we obtain a finite basis for admissible Π_2-rules in $\mathsf{S^2IC}$.

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