The general algebraic framework for Mathematical Fuzzy Logic

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Originating as an attempt to provide solid logical foundations for fuzzy set theory [19], and motivated also by philosophical and computational problems of vagueness and imprecision [16], Mathematical Fuzzy Logic (MFL) has become a significant subfield of mathematical logic [17]. Throughout the years many particular many-valued logics and families of logics have been proposed and investigated by MFL and numerous deep mathematical results have been proven about them (see the three volumes of handbook of MFL [5]). In the early years, the necessay exploratory work of the pionneers resulted naturally in a certain amount of repetition in the papers published on this topic; it was common to encounter articles that studied slightly different logics by repeating the same definitions and essentially obtaining the same results by means of analogous proofs. Therefore, MFL was an area of science screaming for systematization through the development and application of uniform, general, and abstract methods.

Abstract algebraic logic presented itself as the ideal toolbox to rely on; indeed, this general theory is applicable to all non-classical logics and provides an abstract insight into the fundamental (meta)logical properties at play. However, the existing works in that area (summarized in excellent monographs [2,14,15]) did not readily give the desired answers. Despite their many merits, these texts live at a level of abstraction a little too far detached from the intended field of application in MFL. They are indeed great sources of knowledge and inspiration, but there is still a lot of work to be done in order to bring the theory closer to the characteristic particularities of MFL, in particular in first-order logics.

These considerations led us, the authors of this contribution, to writing an extensive series of papers (e.g., [1,3,4,6–8,10–12,18] to name the most important ones) in which we have developed various aspects of the general theory of MFL at different levels of generality and abstraction.

Our first attempt at systematizing this bulk of research was a chapter published in 2011 in the Handbook of Mathematical Fuzzy Logic [9] where we provided rudiments of a well rounded theory constituting solid foundations sufficient (and necessary!) for a rapid development of new particular fuzzy logics demanded by emerging applications. The goal of this talk is to summarize the subsequent 10 years of development and refinements of this theory and present its now matured state of the art as described in our recent monograph [13].

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