# Logics of upsets of De Morgan lattices

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Even a very cursory review of the existing literature on non-classical logics will quickly reveal two facts. Firstly, many of the non-classical logics which have attracted the most attention among the community of algebraic logicians have a conjunction which is interpreted by a binary meet operation in some algebra with a distributive lattice reduct. Secondly, logics with such a lattice conjunction are almost inevitably assumed to satisfy the rule of *adjunction*:

$$x, y \vdash x \wedge y.$$

This rule, together with the rules $x \wedge y \vdash x$ and $x \wedge y \vdash y$, ensures that the designated sets of these logics form lattice filters in some appropriate class of distributive lattice-ordered algebras.

In this contribution, we develop tools which will enable us to study logics with a distributive lattice conjunction where the rule of adjunction fails. In other words, we will be concerned with *logics of upsets*, rather than logics of lattice filters.

As a case study, we shall consider logics determined by a class of matrices of the form $\langle \mathbf{A}, F \rangle$ where $\mathbf{A}$ is a De Morgan lattice and $F$ is an upset of $\mathbf{A}$. However, the results stated below are much more general. The only feature of De Morgan lattices which we use is that they are generated as a quasivariety by a finite algebra, namely the four-element subdirectly-irreducible De Morgan lattice $\mathbf{DM}_1$, and that each prime filter on De Morgan lattice is a homomorphic preimage of a certain prime filter $Q_1$ on $\mathbf{DM}_1$, namely the filter $\{\mathsf{t}, \mathsf{b}\}$.

Logics of filters of De Morgan lattices have in fact recently been studied under the name *super-Belnap logics* [4, 1, 3]. The results presented below can be interpreted as extending the super-Belnap universe to cover natural logics such as Shramko's logic of "anything but falsehood" [5] which do not validate the rule of adjunction but which fit in well with the rest of the super-Belnap family in terms of their motivation. Indeed, extending the notion of a super-Belnap logic to cover such logics was first proposed by Shramko [6].

Our main results are the following two finite basis theorems. Their proofs are constructive: we provide an algorithm which finds the required axiomatizations. The second theorem yields finite Gentzen-style calculi even for logics which have no finite Hilbert-style calculus, such as the extension of Belnap–Dunn logic by the infinite set of rules $(x_1 \wedge \neg x_1) \vee \cdots \vee (x_n \wedge \neg x_n) \vdash y$, which is complete with respect to an eight-element matrix.

**Theorem 1.** *Each logic determined by a finite set of finite matrices of the form $\langle \mathbf{A}, F \rangle$, where $\mathbf{A}$ is a De Morgan lattice and $F$ is a prime upset of $\mathbf{A}$, has a finite Hilbert-style axiomatization.*

**Theorem 2.** *Each logic determined by a finite set of finite matrices of the form $\langle \mathbf{A}, F \rangle$, where $\mathbf{A}$ is a De Morgan lattice and $F$ is a lattice filter of $\mathbf{A}$, has a finite Gentzen-style axiomatization.*

The key tool in proving these theorems will be the notion of an $n$-filter. The theorems will follow easily once we extend basic facts about filters on distributive lattices to $n$-filters.

An upset $F$ of a distributive lattice $\mathbf{A}$ will be called an *$n$-filter*, for $n \geq 1$, if for each non-empty finite $X \subseteq \mathbf{A}$

$$\bigwedge Y \in F \text{ for each } Y \subseteq_n X \implies \bigwedge X \in F,$$

where we use the notation

$$X \subseteq_n Y \iff X \subseteq Y \text{ and } 1 \leq |X| \leq n.$$

We may restrict without loss of generality to $|X| = n+1$ and $|Y| = n$ in this definition. Equivalently, $F$ is an $n$-filter if the matrix $\langle \mathbf{A}, F \rangle$ validates the rule of *$n$-adjunction*:

$$\{\bigwedge_{j \neq i} x_j \mid 1 \leq i \leq n+1\} \vdash x_1 \wedge \cdots \wedge x_{n+1},$$

where $\bigwedge_{j \neq i} x_j$ denotes the submeet of $x_1 \wedge \cdots \wedge x_{n+1}$ obtained by omitting $x_i$. For example, 1-adjunction is the ordinary rule of adjunction, while 2-adjunction is the rule

$$x \wedge y, y \wedge z, z \wedge x \vdash x \wedge y \wedge z.$$

Of course, each $m$-filter is an $n$-filter for $m \leq n$.

Because $n$-filters are closed under arbitrary intersection, we may talk about the $n$-filter $[U]_n$ *generated* by a subset $U$ of $\mathbf{A}$. While understanding filter generation in arbitrary lattices is easy, we only have a good description of $n$-filter generation for $n > 1$ in distributive lattices.

**Lemma 3.** *Let $U$ be an upset of a distributive lattice $\mathbf{A}$. Then $a \in [U]_n$ if and only if there is a non-empty finite set $X \subseteq \mathbf{A}$ such that $\bigwedge Y \in U$ for each $Y \subseteq_n X$ and $\bigwedge X \leq a$.*

Understanding how $n$-filters are generated allows us to prove the following theorem.

**Theorem 4.** *Each $n$-filter on a distributive lattice is an intersection of prime $n$-filters.*

An easy way of constructing $n$-filters is to take the union of a family of at most $n$ filters. This does not suffice to construct all $n$-filters, but it does suffice to construct all prime $n$-filters. Here an upset $U$ is called *prime* if $a \vee b \in U$ implies that either $a \in U$ or $b \in U$.

**Theorem 5.** *Each prime $n$-filter on a distributive lattice is a union of at most $n$ prime filters.*

It remains to describe unions of at most $n$ prime filters as the homomorphic preimages of a certain fixed upset. To this end, the *dual product* construction is useful. Given a family of matrices $\langle \mathbf{A}_i, F_i \rangle$ for $i \in I$, its dual product $\bigotimes_{i \in I} \langle \mathbf{A}_i, F_i \rangle$ is the matrix $\langle \mathbf{A}, F \rangle$ with $\mathbf{A} := \prod_{i \in I} \mathbf{A}_i$ and $F := \bigcup_{i \in I} \pi_i^{-1}[F_i]$, where $\pi \colon \mathbf{A} \to \mathbf{A}_i$ are the projection maps. In other words, a tuple $a \in \mathbf{A}$ is designated in the dual product if and only if some component $a_i \in \mathbf{A}_i$ of this tuple is designated in $\langle \mathbf{A}_i, F_i \rangle$. Let $\langle \mathbf{B}_n, P_n \rangle$ be the $n$-th dual power of the matrix $\langle \mathbf{B}_1, P_1 \rangle$. That is, $a \in P_n$ if and only if $a > \mathsf{f}$ in $\mathbf{B}_n$, where $\mathsf{f}$ denotes the bottom element of $\mathbf{B}_n$.

**Lemma 6.** *An upset $U$ of a distributive lattice is a union of at most $n$ prime filters if and only if it is a homomorphic preimage of the upset $P_n$ of $\mathbf{B}_n$.*

Summing up: $n$-filters on distributive lattices are defined syntactically as upsets which satisfy the rule of $n$-adjunction, but they an also be characterized semantically as the intersections of homomorphic preimages of the prime $n$-filter $P_n \subseteq \mathbf{B}_n$.

This allows us to describe all logics of upsets of distributive lattices, i.e. logics determined by some class of matrices of the form $\langle \mathbf{A}, F \rangle$ where $\mathbf{A}$ is a distributive lattice and $F$ is an upset of $\mathbf{A}$. These are precisely the extensions of the logic $\mathcal{DL}_\infty$ of all upsets of distributive lattices. Let $\mathcal{DL}_n$ be the extension of $\mathcal{DL}_\infty$ by the rule of $n$-adjunction, or equivalently let $\mathcal{DL}_n$ be the logic of all $n$-filters of distributive lattices. It will be convenient to take $\mathbf{B}_0$ to be the trivial lattice, 0-adjunction to be the rule $x \vdash y$, and $P_0$ to be the empty set.

**Theorem 7.** *The logic $\mathcal{DL}_n$ is complete with respect to the matrix $\langle \mathbf{B}_n, P_n \rangle$. Moreover, the logics $\mathcal{DL}_n$ for $n \in \omega$ are the only non-trivial proper extensions of $\mathcal{DL}_\infty$.*

Moving to the setting of De Morgan lattices, much of the above argument remains valid if we replace the prime filter $P_1$ on $\mathbf{B}_1$ by a prime filter $Q_1$ on $\mathbf{DM}_1$. (This filter consists of the top element and one of the fixpoints of negation.) We again define the matrix $\langle \mathbf{DM}_n, Q_n \rangle$ to be the $n$-th dual power of the matrix $\langle \mathbf{DM}_1, Q_1 \rangle$ and obtain the following completeness theorems for the logics $\mathcal{BD}_n$ of $n$-filters of De Morgan lattices, which extend the logic $\mathcal{BD}_\infty$ of all upsets of De Morgan lattices by the rule of $n$-adjunction.

**Theorem 8.** *The logic $\mathbf{DM}_n$ is complete with respect to the matrix $\langle \mathbf{DM}_n, Q_n \rangle$.*

The problem of axiomatizing the logic given by a finite set of prime upsets of De Morgan lattices reduces to the problem of axiomatizing the logic $\mathcal{L}$ given by a set $S$ of submatrices of the finite matrix $\langle \mathbf{DM}_n, Q_n \rangle$ for some $n$: each upset of a finite De Morgan lattice is in fact an $n$-filter for some $n$, and if it is moreover prime, then it is a homomorphic image of $Q_n$. Furthermore, for each submatrix $\langle \mathbf{A}, F \rangle$ of $\langle \mathbf{DM}_n, Q_n \rangle$ there is either a finitary semantic construction of $\langle \mathbf{A}, F \rangle$ in terms of matrices from $S$ witnessing that it is a model of $\mathcal{L}$ or a finitary rule which fails in $\langle \mathbf{A}, F \rangle$ but holds in $\mathcal{L}$. This yields a finite set of finitary rules $R$ such that $\mathcal{L}$ is the smallest extension of $\mathcal{BD}_n$ which validates each rule in $R$ and which is complete with respect to a class of prime upsets. This is equivalent to the claim that $\mathcal{L}$ is axiomatized relative to $\mathcal{BD}_n$ by what we call the *disjunctive variants* of the rules in $R$. This yields a finite Hilbert-style axiomatization for each logic determined by a finite set of prime upsets of De Morgan lattices.

As a concrete application of the algorithm sketched above, we obtain an axiomatization of the logic "anything but falsehood" introduced recently by Shramko [5] as the semantic dual to the logic of "nothing but the truth" introduced by Pietz and Rivieccio [2]. This is the logic determined by the matrix $\langle \mathbf{DM}_1, \{\mathsf{t}, \mathsf{n}, \mathsf{b}\} \rangle$, where $\mathsf{n}$ and $\mathsf{b}$ are the two fixpoints of negation in $\mathbf{DM}_1$ and $\mathsf{t}$ is the top element. The last rule in the axiomatization below is what we call the disjunctive variant of the rule $x, \neg x \vdash x \wedge \neg x$.

**Theorem 9.** *The logic of the structure $\langle \mathbf{DM}_1, \{\mathsf{t}, \mathsf{n}, \mathsf{b}\} \rangle$ is the extension of $\mathcal{BD}_\infty$ by the 2-adjunction rule, the law of the excluded middle $\emptyset \vdash x \vee \neg x$, and the rule $x \vee y, \neg x \vee y \vdash (x \wedge \neg x) \vee y$.*

To obtain the following theorem, it now suffices to observe that a finitary extension $\mathcal{L}$ of $\mathcal{BD}_\infty$ is complete with respect to some class of matrices of the form $\langle \mathbf{A}, F \rangle$ where $F$ is a prime upset if and only if it satisfies the *proof by cases property (PCP)*:

$$\Gamma, \varphi_1 \vee \varphi_2 \vdash_{\mathcal{L}} \psi \iff \Gamma, \varphi_1 \vdash_{\mathcal{L}} \psi \text{ and } \Gamma, \varphi_2 \vdash_{\mathcal{L}} \psi.$$

**Theorem 10.** *The following are equivalent for each extension $\mathcal{L}$ of $\mathcal{BD}_\infty$:*

*(i)* *$\mathcal{L}$ is a finitary extension of $\mathcal{BD}_n$ with the PCP,*

*(ii)* *$\mathcal{L}$ is complete with respect to some set of substructures of $\langle \mathbf{DM}_n, Q_n \rangle$,*

*(iii)* *$\mathcal{L}$ is complete with respect to some finite set of finite structures of the form $\langle \mathbf{L}, F \rangle$ where $\mathbf{L}$ is a De Morgan lattice and $F$ is a prime $n$-filter of $\mathbf{L}$.*

*Some such $n$ exists whenever $\mathcal{L}$ has the PCP and is complete w.r.t. a finite set of finite matrices.*

The case of logics determined by a finite set of filters (rather than prime upsets) of De Morgan lattices admits an analogous analysis, but we need to consider $n$-prime filters (rather than prime

$n$-filters). A filter $F$ on a distributive lattice $\mathbf{A}$ wil be called *$n$-prime* if it is a meet $n$-prime element of the lattice of all filters on $\mathbf{A}$, i.e. if for each non-empty finite family of filters $\mathcal{F}$ on $\mathbf{A}$

$$\bigcap \mathcal{F} \subseteq F \implies \bigcap \mathcal{G} \subseteq F \text{ for some } \mathcal{G} \subseteq_n \mathcal{F}.$$

Equivalently, $n$-prime filters are precisely the complements of prime $n$-ideals.

A finitary extension $\mathcal{L}$ of $\mathcal{BD}_1$ is complete with respect to a class of $n$-prime filters if and only if it satisfies what we call the *$n$-proof by cases property ($n$-PCP)*:

$$\Gamma, \bigvee_{j \neq 1} \varphi_j \vdash_{\mathcal{L}} \psi \text{ and } \dots \text{ and } \Gamma, \bigvee_{j \neq n+1} \varphi_j \vdash_{\mathcal{L}} \psi \implies \Gamma, \varphi_1 \vee \dots \vee \varphi_{n+1} \vdash_{\mathcal{L}} \psi.$$

In particular, the 2-PCP states the following:

$$\Gamma, \varphi_1 \vee \varphi_2 \vdash_{\mathcal{L}} \psi \text{ and } \Gamma, \varphi_2 \vee \varphi_3 \vdash_{\mathcal{L}} \psi \text{ and } \Gamma, \varphi_3 \vee \varphi_1 \vdash_{\mathcal{L}} \psi \implies \Gamma, \varphi_1 \vee \varphi_2 \vee \varphi_3 \vdash_{\mathcal{L}} \psi.$$

We now obtain the following theorem in a manner entirely analogous to the previous one.

**Theorem 11.** *The following are equivalent for each extension $\mathcal{L}$ of $\mathcal{BD}_1$:*

*(i) $\mathcal{L}$ is a finitary and enjoys the $n$-PCP,*

*(ii) $\mathcal{L}$ is complete with respect to some set of substructures of $(\mathbb{DM}_1)^n$,*

*(iii) $\mathcal{L}$ is complete with respect to some finite set of finite structures of the form $\langle \mathbf{L}, F \rangle$ where $\mathbf{L}$ is a De Morgan lattice and $F$ is an $n$-prime upset of $\mathbf{L}$.*

*Some such $n$ exists whenever $\mathcal{L}$ is complete w.r.t. a finite set of finite matrices.*

In this case, $\mathcal{L}$ is the smallest logic satisfying the $n$-PCP and a certain finite set of finitary rules $R$. This description of $\mathcal{L}$ cannot, in general, be transformed into a finite Hilbert-style axiomatization of $\mathcal{L}$: some logics determined by a filter on a finite De Morgan lattice do not admit any finite Hilbert-style axiomatization. We do, however, obtain a finite Gentzen-style axiomatization of $\mathcal{L}$, the key Gentzen-style rule being the $n$-PCP.

# References


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