Logics of upsets of De Morgan lattices

ADAM PŘENOSIL
Università degli Studi di Cagliari, Italy
adam.prenosil@gmail.com

Even a very cursory review of the existing literature on non-classical logics will quickly reveal two facts. Firstly, many of the non-classical logics which have attracted the most attention among the community of algebraic logicians have a conjunction which is interpreted by a binary meet operation in some algebra with a distributive lattice reduct. Secondly, logics with such a lattice conjunction are almost inevitably assumed to satisfy the rule of adjunction:

$$x, y \vdash x \land y.$$  

This rule, together with the rules $x \land y \vdash x$ and $x \land y \vdash y$, ensures that the designated sets of these logics form lattice filters in some appropriate class of distributive lattice-ordered algebras.

In this contribution, we develop tools which will enable us to study logics with a distributive lattice conjunction where the rule of adjunction fails. In other words, we will be concerned with logics of upsets, rather than logics of lattice filters.

As a case study, we shall consider logics determined by a class of matrices of the form $\langle A, F \rangle$ where $A$ is a De Morgan lattice and $F$ is an upset of $A$. However, the results stated below are much more general. The only feature of De Morgan lattices which we use is that they are generated as a quasivariety by a finite algebra, namely the four-element subdirectly-irreducible De Morgan lattice $DM_1$, and that each prime filter on De Morgan lattice is a homomorphic preimage of a certain prime filter $Q_1$ on $DM_1$, namely the filter $\{t, b\}$.

Logics of filters of De Morgan lattices have in fact recently been studied under the name super-Belnap logics [4, 1, 3]. The results presented below can be interpreted as extending the super-Belnap universe to cover natural logics such as Shramko’s logic of “anything but falsehood” [5] which do not validate the rule of adjunction but which fit in well with the rest of the super-Belnap family in terms of their motivation. Indeed, extending the notion of a super-Belnap logic to cover such logics was first proposed by Shramko [6].

Our main results are the following two finite basis theorems. Their proofs are constructive: we provide an algorithm which finds the required axiomatizations. The second theorem yields finite Gentzen-style calculi even for logics which have no finite Hilbert-style calculus, such as the extension of Belnap–Dunn logic by the infinite set of rules $(x_1 \land \neg x_1) \lor \cdots \lor (x_n \land \neg x_n) \vdash y$, which is complete with respect to an eight-element matrix.

**Theorem 1.** Each logic determined by a finite set of finite matrices of the form $\langle A, F \rangle$, where $A$ is a De Morgan lattice and $F$ is an upset of $A$, has a finite Hilbert-style axiomatization.

**Theorem 2.** Each logic determined by a finite set of finite matrices of the form $\langle A, F \rangle$, where $A$ is a De Morgan lattice and $F$ is a lattice filter of $A$, has a finite Gentzen-style axiomatization.

The key tool in proving these theorems will be the notion of an $n$-filter. The theorems will follow easily once we extend basic facts about filters on distributive lattices to $n$-filters.

An upset $F$ of a distributive lattice $A$ will be called an $n$-filter, for $n \geq 1$, if for each non-empty finite $X \subseteq A$

$$\bigwedge Y \in F \text{ for each } Y \subseteq_n X \implies \bigwedge X \in F,$$
where we use the notation

\[ X \subseteq_n Y \iff X \subseteq Y \text{ and } 1 \leq |X| \leq n. \]

We may restrict without loss of generality to \(|X| = n + 1\) and \(|Y| = n\) in this definition. Equivalently, \( F \) is an \( n \)-filter if the matrix \( \langle A, F \rangle \) validates the rule of \( n \)-adjunction:

\[ \{ \bigwedge_{j \neq i} x_j \mid 1 \leq i \leq n + 1 \} \vdash x_1 \land \cdots \land x_{n+1}, \]

where \( \bigwedge_{j \neq i} x_j \) denotes the submeet of \( x_1 \land \cdots \land x_{n+1} \) obtained by omitting \( x_i \). For example, 1-adjunction is the ordinary rule of adjunction, while 2-adjunction is the rule

\[ x \land y, y \land z, z \land x \vdash x \land y \land z. \]

Of course, each \( m \)-filter is an \( n \)-filter for \( m \leq n \).

Because \( n \)-filters are closed under arbitrary intersection, we may talk about the \( n \)-filter \( [U]_n \) generated by a subset \( U \) of \( A \). While understanding filter generation in arbitrary lattices is easy, we only have a good description of \( n \)-filter generation for \( n > 1 \) in distributive lattices.

**Lemma 3.** Let \( U \) be an upset of a distributive lattice \( A \). Then \( a \in [U]_n \) if and only if there is a non-empty finite set \( X \subseteq A \) such that \( \bigwedge Y \in U \) for each \( Y \subseteq_n X \) and \( \bigwedge X \leq a \).

Understanding how \( n \)-filters are generated allows us to prove the following theorem.

**Theorem 4.** Each \( n \)-filter on a distributive lattice is an intersection of prime \( n \)-filters.

An easy way of constructing \( n \)-filters is to take the union of a family of at most \( n \) filters. This does not suffice to construct all \( n \)-filters, but it does suffice to construct all prime \( n \)-filters. Here an upset \( U \) is called prime if \( a \lor b \in U \) implies that either \( a \in U \) or \( b \in U \).

**Theorem 5.** Each prime \( n \)-filter on a distributive lattice is a union of at most \( n \) prime filters.

It remains to describe unions of at most \( n \) prime filters as the homomorphic preimages of a certain fixed upset. To this end, the dual product construction is useful. Given a family of matrices \( \langle A_i, F_i \rangle \) for \( i \in I \), its dual product \( \bigotimes_{i \in I} \langle A_i, F_i \rangle \) is the matrix \( \langle A, F \rangle \) with \( A := \prod_{i \in I} A_i \) and \( F := \bigcup_{i \in I} \pi_i^{-1}[F_i] \), where \( \pi: A \to A_i \) are the projection maps. In other words, a tuple \( a \in A \) is designated in the dual product if and only if some component \( a_i \in A_i \) of this tuple is designated in \( \langle A_i, F_i \rangle \). Let \( \langle B_n, P_n \rangle \) be the \( n \)-th dual power of the matrix \( \langle B_1, P_1 \rangle \). That is, \( a \in P_n \) if and only if \( a > f \) in \( B_n \), where \( f \) denotes the bottom element of \( B_n \).

**Lemma 6.** An upset \( U \) of a distributive lattice is a union of at most \( n \) prime filters if and only if it is a homomorphic preimage of the upset \( P_n \) of \( B_n \).

Summing up: \( n \)-filters on distributive lattices are defined syntactically as upsets which satisfy the rule of \( n \)-adjunction, but they also be characterized semantically as the intersections of homomorphic preimages of the prime \( n \)-filter \( P_n \subseteq B_n \).

This allows us to describe all logics of upsets of distributive lattices, i.e. logics determined by some class of matrices of the form \( \langle A, F \rangle \) where \( A \) is a distributive lattice and \( F \) is an upset of \( A \). These are precisely the extensions of the logic \( \mathcal{DL}_\infty \) of all upsets of distributive lattices. Let \( \mathcal{DL}_n \) be the extension of \( \mathcal{DL}_\infty \) by the rule of \( n \)-adjunction, or equivalently let \( \mathcal{DL}_n \) be the logic of all \( n \)-filters of distributive lattices. It will be convenient to take \( B_0 \) to be the trivial lattice, 0-adjunction to be the rule \( x \vdash y \), and \( P_0 \) to be the empty set.
Theorem 7. The logic $\mathcal{DL}_n$ is complete with respect to the matrix $\langle B_n, P_n \rangle$. Moreover, the logics $\mathcal{DL}_n$ for $n \in \omega$ are the only non-trivial proper extensions of $\mathcal{DL}_\infty$.

Moving the setting of De Morgan lattices, much of the above argument remains valid if we replace the prime filter $P_1$ on $B_1$ by a prime filter $Q_1$ on $\mathcal{DM}_1$. (This filter consists of the top element and one of the fixpoints of negation.) We again define the matrix $\langle DM_n, Q_n \rangle$ to be the $n$-th dual power of the matrix $\langle DM_1, Q_1 \rangle$ and obtain the following completeness theorems for the logics $BD_n$ of $n$-filters of De Morgan lattices, which extend the logic $BD_\infty$ of all upsets of De Morgan lattices by the rule of $n$-adjunction.

Theorem 8. The logic $\mathcal{DM}_n$ is complete with respect to the matrix $\langle DM_n, Q_n \rangle$.

The problem of axiomatizing the logic given by a finite set of prime upsets of De Morgan lattices reduces to the problem of axiomatizing the logic $\mathcal{L}$ given by a set $S$ of submatrices of the finite matrix $\langle DM_n, Q_n \rangle$ for some $n$: each upset of a finite De Morgan lattice is in fact an $n$-filter for some $n$, and if it is moreover prime, then it is a homomorphic image of $Q_n$. Furthermore, for each submatrix $\langle A, F \rangle$ of $\langle DM_n, Q_n \rangle$ there is either a finitary semantic construction of $\langle A, F \rangle$ in terms of matrices from $S$ witnessing that it is a model of $\mathcal{L}$ or a finitary rule which fails in $\langle A, F \rangle$ but holds in $\mathcal{L}$. This yields a finite set of finitary rules $R$ such that $\mathcal{L}$ is the smallest extension of $BD_n$ which validates each rule in $R$ and which is complete with respect to a class of prime upsets. This is equivalent to the claim that $\mathcal{L}$ is axiomatized relative to $BD_n$ by what we call the disjunctive variants of the rules in $R$. This yields a finite Hilbert-style axiomatization for each logic determined by a finite set of prime upsets of De Morgan lattices.

As a concrete application of the algorithm sketched above, we obtain an axiomatization of the logic “anything but falsehood” introduced recently by Shramko [5] as the semantic dual to the logic of “nothing but the truth” introduced by Pietz and Rivieccio [2]. This is the logic determined by the matrix $\langle DM_1, \{t, n, b\} \rangle$, where $n$ and $b$ are the two fixpoints of negation in $DM_1$ and $t$ is the top element. The last rule in the axiomatization below is what we call the disjunctive variant of the rule $x, \neg x \vdash x \land \neg x$.

Theorem 9. The logic of the structure $\langle DM_1, \{t, n, b\} \rangle$ is the extension of $BD_\infty$ by the 2-adjunction rule, the law of the excluded middle $\emptyset \vdash x \lor \neg x$, and the rule $x \lor y, \neg x \lor y \vdash (x \land \neg x) \lor y$.

To obtain the following theorem, it now suffices to observe that a finitary extension $\mathcal{L}$ of $BD_\infty$ is complete with respect to some class of matrices of the form $\langle A, F \rangle$ where $F$ is a prime upset if and only if it satisfies the proof by cases property (PCP):

$$\Gamma, \varphi_1 \lor \varphi_2 \vdash_C \psi \iff \Gamma, \varphi_1 \vdash_C \psi \text{ and } \Gamma, \varphi_2 \vdash_C \psi.$$ 

Theorem 10. The following are equivalent for each extension $\mathcal{L}$ of $BD_\infty$:

(i) $\mathcal{L}$ is a finitary extension of $BD_n$ with the PCP,

(ii) $\mathcal{L}$ is complete with respect to some set of substructures of $\langle DM_n, Q_n \rangle$,

(iii) $\mathcal{L}$ is complete with respect to some finite set of finite structures of the form $\langle L, F \rangle$ where $L$ is a De Morgan lattice and $F$ is a prime $n$-filter of $L$.

Some such $n$ exists whenever $\mathcal{L}$ has the PCP and is complete w.r.t. a finite set of finite matrices.

The case of logics determined by a finite set of filters (rather than prime upsets) of De Morgan lattices admits an analogous analysis, but we need to consider $n$-prime filters (rather than prime
n-filters). A filter $F$ on a distributive lattice $A$ will be called $n$-prime if it is a meet $n$-prime element of the lattice of all filters on $A$, i.e. if for each non-empty finite family of filters $\mathcal{F}$ on $A$

$$\bigcap \mathcal{F} \subseteq F \implies \bigcap \mathcal{G} \subseteq F \text{ for some } \mathcal{G} \subseteq_n \mathcal{F}.$$ 

Equivalently, $n$-prime filters are precisely the complements of prime $n$-ideals.

A finitary extension $\mathcal{L}$ of $\mathcal{BD}_1$ is complete with respect to a class of $n$-prime filters if and only if it satisfies what we call the $n$-proof by cases property ($n$-PCP):

$$\Gamma, \bigvee_{j \neq 1} \varphi_j \vdash_L \psi \text{ and } \ldots \text{ and } \Gamma, \bigvee_{j \neq n+1} \varphi_j \vdash_L \psi \implies \Gamma, \varphi_1 \lor \cdots \lor \varphi_{n+1} \vdash_L \psi.$$ 

In particular, the 2-PCP states the following:

$$\Gamma, \varphi_1 \lor \varphi_2 \vdash_L \psi \text{ and } \Gamma, \varphi_2 \lor \varphi_3 \vdash_L \psi \text{ and } \Gamma, \varphi_3 \lor \varphi_1 \vdash_L \psi \implies \Gamma, \varphi_1 \lor \varphi_2 \lor \varphi_3 \vdash_L \psi.$$ 

We now obtain the following theorem in a manner entirely analogous to the previous one.

**Theorem 11.** The following are equivalent for each extension $\mathcal{L}$ of $\mathcal{BD}_1$:

(i) $\mathcal{L}$ is a finitary and enjoys the $n$-PCP,

(ii) $\mathcal{L}$ is complete with respect to some set of substructures of $(\mathcal{DM}_1)^n$,

(iii) $\mathcal{L}$ is complete with respect to some finite set of finite structures of the form $(L, F)$ where $L$ is a De Morgan lattice and $F$ is an $n$-prime upset of $L$.

Some such $n$ exists whenever $\mathcal{L}$ is complete w.r.t. a finite set of finite matrices.

In this case, $\mathcal{L}$ is the smallest logic satisfying the $n$-PCP and a certain finite set of finitary rules $R$. This description of $\mathcal{L}$ cannot, in general, be transformed into a finite Hilbert-style axiomatization of $\mathcal{L}$: some logics determined by a filter on a finite De Morgan lattice do not admit any finite Hilbert-style axiomatization. We do, however, obtain a finite Gentzen-style axiomatization of $\mathcal{L}$, the key Gentzen-style rule being the $n$-PCP.

**References**


