Positive (Modal) Logic Beyond Distributivity

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1 Introduction

Duality between modal algebras and modal spaces on the one hand and Heyting algebras and Esakia spaces on the other have been central to the study of modal and intermediate logics [4, 6]. Many important results such as Sahlqvist canonicity and correspondence use duality [13]. In [5], duality between modal algebras and modal spaces is extended to modal distributive lattices (i.e. with distributive lattices taking the role of Boolean algebras) and modal Priestley spaces. Among other things, this led to Sahlqvist theory for positive distributive modal logic.

When the algebraic side of a duality is based on Boolean algebras or distributive lattices, in the spatial side of the duality one works with the space of prime filters of a given lattice. This no longer works for non-distributive lattices. There have been many attempts to extend a duality for Boolean algebras and distributive lattices to the setting of all lattices, e.g. by Urquhart, Hartonas, Gehrke and van Gool, and Goldblatt (we skip the references for lack of space). While this has proven a fruitful and interesting approach, it is quite different from known dualities for propositional logics such as Stone and Priestley duality. As a consequence, it can be difficult to modify existing tools and techniques from other propositional bases for these dualities.

An approach towards duality for non-distributive meet-semilattices was developed by Hofmann, Mislove and Stralka (HMS) [10], along the same lines of the proof of the Van Kampen-Pontryagin duality for locally compact abelian groups. This was later modified to a duality for lattices by Jipsen and Moshier [12]. In HMS duality the dual space is based not on prime filters, but all (proper) filters of a lattice. This is closely related to the possibility semantics of modal logic (Holiday) and to choice-free duality for Boolean algebras (N. Bezhanishvili and Holliday), where again one works with the space of all proper filters. Such an approach was also developed for ortholattices by Goldblatt [9] and later extended by Bimbo [3].

Here we restrict HMS duality to a Stone type duality for lattices, which in turn we extend to modal lattices. As a result we obtain a new Kripke style semantics for non-distributive positive logic, and Sahlqvist correspondence and completeness results for (modal) non-distributive positive logic with their Kripke-style semantics. We also obtain an alternative proof of Bakers and Hales' result [1] that every variety of lattices is closed under ideal completions and extend this result to varieties of modal lattices. This abstract is based on [7, 2].

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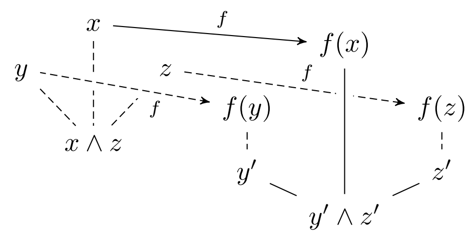
2 Non-distributive positive logic

Let $\mathbf{L}$ be the language of positive logic. We investigate the logic $\mathcal{L}$ consisting of consequence pairs, whose algebraic semantics are (not necessarily distributive) lattices. From a semantic point of view, the move from distributive to non-distributive positive logic is given by:

(Step 1) replace "poset" with "meet-semilattice;"

(Step 2) replace "upset" with "filter."

2.1 Definition. A *lattice Kripke frame* or *L-frame* is a meet-semilattice $(X, \wedge)$. An *L-morphism* from $(X, \wedge)$ to $(X', \wedge')$ is a meet-preserving function $f : (X, \wedge) \to (X', \wedge')$ that satisfies for all $x \in X$ and $y', z' \in X'$: if $y' \wedge z' \leq f(x)$ then there exist $y, z \in X$ such that $y' \leq' f(y)$ and $z' \leq' f(z)$ and $y \wedge z \leq x$ (see figure on the right).



An *L-model* $(X, \wedge, V)$ is an L-frame with a *valuation* that assigns to each proposition letter a filter of $(X, \wedge)$. The interpretation $[\![\phi]\!]$ of $\phi \in \mathbf{L}$ is given by

$$[\![\top]\!] = X \qquad [\![\bot]\!] = \emptyset$$
$$[\![p]\!] = V(p) \qquad [\![\phi \wedge \psi]\!] = [\![\phi]\!] \cap [\![\psi]\!]$$
$$[\![\phi \vee \psi]\!] = [\![\phi]\!] \cup [\![\psi]\!] \cup \uparrow\{x \wedge y \mid x \in [\![\phi]\!], y \in [\![\psi]\!]\}$$

It can be shown that the interpretation of every formula is a filter. We say that a frame $(X, \wedge)$ *validates* the consequence pair $\phi \trianglelefteq \psi$ if $[\![\phi]\!] \subseteq [\![\psi]\!]$ for every model based on it, and write $(X, \wedge) \Vdash \phi \trianglelefteq \psi$. We obtain a duality for the category Lat of lattices by restricting HMS duality.

2.2 Definition. An *HMS space* is a tuple $(X, \wedge, \tau)$ such that $(X, \wedge)$ is a meet-semilattice and (X, τ) is a compact topological space, which additionally satisfies the *HMS separation axiom*:

$$\text{if } x \not\leq y \text{ then there exists a clopen filter } a \text{ such that } x \in a \text{ and } y \notin a.$$

(Here $\leq$ is the order induced by $\wedge$.) An HMS space is called an *L-space* if for every pair of clopen filters a, b, the filter $a \curlyvee b := a \cup b \cup \uparrow\{x \wedge y \mid x \in a, y \in b\}$ is clopen as well.

We write HMS for the category of HMS spaces and continuous meet-semilattice morphisms, and LSpace for the category of L-spaces and continuous L-morphisms.

2.3 Theorem. *We have* $\mathsf{MSL} \equiv^{\text{op}} \mathsf{HMS}$ *[10], and this restricts to* $\mathsf{Lat} \equiv^{\text{op}} \mathsf{LSpace}$*.*

Here MSL denotes the category of meet-semilattices. Clearly, every L-space $\mathbb{X}$ has an underlying L-frame, denoted by $\kappa\mathbb{X}$. A *clopen valuation* for an L-space is a valuation that assigns to each proposition letter a clopen filter. This gives rise to completeness as usual. Using standard techniques of modal logic (see e.g. [4, Section 3.6]), we obtain the following Sahlqvist results.

2.4 Theorem. *Let* $\phi \trianglelefteq \psi$ *be a consequence pair of* $\mathbf{L}$*-formulae.*

1. $\psi \trianglelefteq \chi$ *locally corresponds to a first-order formula with one free variable.*
2. *For every L-space* $\mathbb{X}$*, if* $\mathbb{X} \Vdash \phi \trianglelefteq \psi$ *then* $\kappa\mathbb{X} \Vdash \phi \trianglelefteq \psi$*.*
3. *If* Γ *is a set of consequence pairs, then* $\mathbf{L}(\Gamma)$ *is sound and complete with respect to the class of L-frames validating all consequence pairs in* Γ*.*

The duality for the Lat gives rise to a new type of lattice completion. We define the F^2-*completion* of a lattice L to be the lattice of all filters of the L-space dual to L. As a consequence of Theorem 3.7 we get the following analogue of [1, Theorem B]:

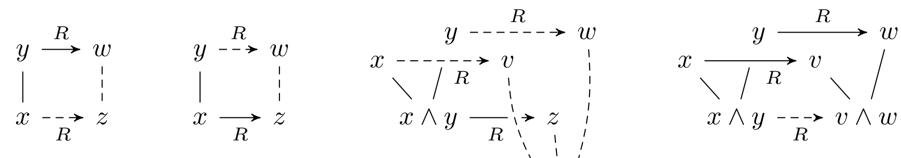
2.5 Theorem. *Every variety of lattices is closed under taking* F^2*-completions.*

3 Modal lattices

We extend the logic from above with modal operators $\Box$ and $\Diamond$. We leave the precise definition of the resulting logic $\mathcal{L}_{\Box\Diamond}$ implicit, and instead give its algebraic semantics in Definition 3.3 below. As a starting point we extend L-frames with an additional relation (used to interpret the modalities), and we stipulate conditions ensuring that every formula is interpreted as a filter.

3.1 Definition. A *modal L-frame* is a tuple $(X, \wedge, R)$ where $(X, \wedge)$ is an L-frame (with induced order $\leq$) and R is a binary relation on X such that:

1. If $x \leq y$ and yRz then there exists a $w \in X$ such that xRw and $w \leq y$;
2. If $x \leq y$ and xRw then there exists a $z \in X$ such that yRz and $w \leq z$;
3. If $(x \wedge y)Rz$ then there exist $v, w \in X$ such that xRv and yRw and $v \wedge w \leq z$;
4. If xRv and yRw then $(x \wedge y)R(v \wedge w)$;
5. For all $x \in X$ there exists an $y \in X$ such that xRy.



(1) (2) (3) (4)

A *bounded L-morphism* from $(X, \wedge, R)$ to $(X', \wedge', R')$ is a function $f : X \to X'$ such that $f : (X, \wedge) \to (X', \wedge')$ is an L-morphism and for all $x, y \in X$ and $z' \in X'$:

1. If xRy then $f(x)R'f(y)$;
2. If $f(x)R'z'$ then there exists a $z \in X$ such that xRz and $f(z) \leq z'$;
3. If $f(x)R'z'$ then there exists a $w \in X$ such that xRz and $z' \leq' f(w)$.

A *modal L-model* is a a modal L-frame with a valuation V that assigns to each proposition letter a filter of $(X, \wedge)$. Propositional connectives are interpreted as in Definition 3.1, and

$$\llbracket \Box\phi \rrbracket = \{x \in X \mid \forall y \in X, xRy \text{ implies } \mathfrak{M}, y \Vdash \phi\}$$
$$\llbracket \Diamond\phi \rrbracket = \{x \in X \mid \exists y \in X \text{ such that } xRy \text{ and } \mathfrak{M}, y \Vdash \phi\}$$

Satisfaction and validity of formulae and consequence pairs are defined as expected.

3.2 Lemma. *The following modal consequence pairs are valid in all modal L-frames:*

$\top \trianglelefteq \Box\top$	$\top \trianglelefteq \Diamond\top$	$\Diamond\bot \trianglelefteq \bot$	*(top and bottom)*
$\Box(p \wedge q) \trianglelefteq \Box p \wedge \Box q$	$\Diamond p \trianglelefteq \Diamond(p \vee q)$		*(monotonicity)*
$\Box p \wedge \Box q \trianglelefteq \Box(p \wedge q)$	$\Diamond p \wedge \Box q \trianglelefteq \Diamond(p \wedge q)$		*(normality and duality)*

3.3 Definition. A *modal lattice* is a tuple $(A, \Box, \Diamond)$ where A is a lattice and $\Box, \Diamond : A \to A$ are maps satisfying the inequalities from Lemma 3.2, with p and q ranging over A and "$\trianglelefteq$" replaced with "$\leq$." With $\Box$- and $\Diamond$-preserving lattice homomorphisms they form the category MLat.

Indeed, $\Diamond$ is not necessarily normal. This resembles the modal extension of intuitionistic logic studied by Kojima [11]. This need not worry us: normality of $\Diamond$ is a Sahlqvist consequence pair, so we can use the results below to restrict to the "fully normal" case. Besides, we have to add seriality ($\top \trianglelefteq \Diamond\top$) because our joins can no longer adequately describe the connection between $\Box$ and $\Diamond$. We obtain a duality for modal lattices by means of L-spaces with relations.

3.4 Definition. A *modal L-space* is a tuple $\mathbb{X} = (X, \wedge, \tau, R)$ such that:

1. $(X, \wedge, \tau)$ is an L-space, R is a binary relation on X, and each $x \in X$ has an R-successor;
2. If a is a clopen filter, then so are $[R]a := \{x \in X \mid R[x] \subseteq a\}$ and $\langle R \rangle a := \{x \in X \mid R[x] \cap a \neq \emptyset\}$;
3. We have xRy iff for all $a \in \mathscr{F}_{clp}\mathbb{X}$, $x \in [R]a$ implies $y \in a$, and $y \in a$ implies $x \in \langle R \rangle a$.

Then it can be shown that $(X, \wedge, R)$ is a modal L-frame. With continuous bounded L-morphisms they form the category $\mathsf{MLSpace}$.

3.5 Theorem. *The duality between* Lat *and* LSpace *lifts to a duality* $\mathsf{MLat} \equiv^{\text{op}} \mathsf{MLSpace}$.

Using standard techniques of modal logic we obtain the following Sahlqvist results.

3.6 Definition. A *boxed atom* is a formula of the form $\Box \cdots \Box p$, with p a proposition letter. A *Sahlqvist antecedent* is a formula made from boxed atoms, $\top$ and $\bot$ by freely using $\wedge$, $\vee$ and $\Diamond$. A *Sahlqvist consequence pair* is a consequence pair $\phi \trianglelefteq \psi$ where ϕ is a Sahlqvist antecedent.

3.7 Theorem. *Let $\phi \trianglelefteq \psi$ be a Sahlqvist consequence pair of $\mathbf{L}_{\Box\Diamond}$-formulae.*

1. *$\psi \trianglelefteq \chi$ locally corresponds to a first-order formula with one free variable.*
2. *For every modal L-space $\mathbb{X}$, if $\mathbb{X} \Vdash \phi \trianglelefteq \psi$ then $\kappa\mathbb{X} \Vdash \phi \trianglelefteq \psi$.*
3. *If Γ is a set of Sahlqvist consequence pairs, then $\mathbf{L}_{\Box\Diamond}(\Gamma)$ is sound and complete with respect to the class of L-frames validating all consequence pairs in Γ.*

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