Abstract Model and Deduction System for Logic of Multiple Agent in Quantum Physics

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Quantum logic (QL) has been studied to handle the strange propositions of quantum physics. Moreover, numerous types of logic and structures have been proposed to represent and analyze these propositions [8] [9]. In particular, logic based on *orthomodular lattices*, namely, *orthomodular logic* (OML), has been studied since 1936, proposed by Birkhoff and Von Neumann [7] [11]. An orthomodular lattice is related to the closed subspaces of a Hilbert space, which is a state space of a particle in quantum physics. Instead of these lattices, the *Kripke model* (possible world model) of OML can be used, which is called the *orthomodular-model* (OM-model) [9] [12] [13] . Intuitively, each possible world of an OM-model expresses a one-dimensional subspace of a Hilbert space, corresponding to a *quantum state*.

To treat an agent's *knowledge* in quantum mechanics, some studies combine *epistemic logic* (EL) with QL. EL is a field of modal logic that treats the proposition of an agent's knowledge. In the Kripke model of EL, the *indistinguishability of states* is used to express knowledge. That is, if a formula ϕ is true at all states that are indistinguishable from the current state for agent i, then agent i knows that ϕ is true. Furthermore, *dynamic EL* (DEL) has been studied to handle the transitions of knowledge. In general, *public announcement logic* (PAL) is treated as the most basic and simple logic in DEL. Basic PAL includes only two types of modal symbols: the symbols for knowledge K_i of individual agents and the symbol [ ] for public announcements. $[\phi]\psi$ can be read as "After a public announcement ϕ, ψ is true." For more details of DEL, see [10]. Ref [5] and [6] can be cited as one of the studies of logic that deal with the concept of knowledge with quantum physics. In these studies, the models which incorporate specific *quantum information* concepts were used. Ref [3] and [4] can be cited as studies of knowledge with more general concepts of quantum physics.

Although knowledge in quantum mechanics has been analyzed in some directions in logic, *abstract* model for this field wasn't much discussed, and *deduction systems* are not well constructed. That is, in general, QL has been developed using two primary methods. The first method is research using models that can express almost all properties of Hilbert spaces. In this context, the Hilbert space is often employed as a model. The second method is research using a simple model that uses only essential parts of a Hilbert space. Studies using orthomodular lattices formed by observational propositions of a Hilbert space belong to this category. The two methods have their advantages and disadvantages. The former method is suitable for detailed and diverse analysis of quantum mechanics because it can express almost all propositions for the states or values of physical quantities in quantum mechanics. However, it has the disadvantage that logical analysis is difficult because logical symbols and models become quite complex. In the latter method, although detailed analysis is impossible, essential properties can be abstractly treated. Further, because simple logical symbols and models are used, it is easy to perform logical analysis and comparison with other logic.

The former method is extremely common when considering propositions about complex notions in quantum mechanics such as agent's knowledge. Especially, to date, there are few logical analyses of knowledge of *multiple* agents (with multiple particles) using an abstract and simple model. One of the reasons is related to a problem using orthomodular lattices. An orthomodular lattice L can be developed by extracting the concept of the closed subspace of a Hilbert space H. If the state space of one particle is H, the state space of two particles is represented by $H \otimes H$, where $\otimes$ denotes the tensor products of spaces. However, intuitively, the tensor product of lattices $L \otimes L$ does not correspond to $H \otimes H$. The tensor products of orthomodular lattices cannot represent a linear combination in a vector space. For example, assume that H is a 2D Hilbert space. Then, $c(|0> \otimes|0> + |1> \otimes|1>)$ is a 1D closed subspace of H. However, an element corresponding to this space is not included in $L \otimes L$, i.e., $L \otimes L$ includes only elements corresponding to the states represented by the multiplication, such as $c|0> \otimes|0>$ or $c|1> \otimes|1>$). Therefore, when handling multiple particles, using the tensor product of orthomodular lattices does not include essential elements such as entanglement in the model [1]. This situation is the same even when using the OM-model.

The situation where multiple agents have their particles is common in quantum mechanics. Therefore, it is meaningful to develop a method that can abstractly discuss propositions in such situations. In this study, we propose some methods and models to overcome the above problem and construct and analyze new logic for knowledge of multiple agents or multiple particles in quantum mechanics.

As a new logic, MDEQL (Multi-particle dynamic epistemic quantum logic) is constructed and discussed. It is desirable to avoid the models that introduce the concept of certain notions of Hilbert space concretely. Therefore, for the basic model of MDEQL, OM-model is adopted the same as OML. By using the OML model and language almost as they are, it becomes easier to analyze and prove the theorem. Then, we limit models to those that satisfy important conditions of a tensor product Hilbert space, i.e., intuitively, it is assumed that a model already corresponds to the tensor product $H \otimes H$of Hilbert spaces, and several properties of individual Hilbert spaces H are represented by additional conditions. This method avoids the above-mentioned deficiencies in developing a tensor product model from models. Based on these models, the models of MDEQL are defined by adding modality relations of knowledge.

The language for MDEQL is defined as follows. We index propositional variables into multiple classes to indicate which particle's proposition each propositional variable represents. Such an expression method is often used [2]. Furthermore, for technical reasons, formulas are defined in two parts. One corresponds to the language of OML and the other corresponds to formulas for expressing knowledge.

q-formula $A ::= p_i \mid\ \sim A \mid A \wedge A$

g-formula $\phi ::= A \mid \neg\phi \mid \phi \wedge \phi \mid \mathsf{K}_i\phi \mid [A]\phi$

$\sim$ is quantum negation, whereas $\neg$ is classical negation. Only q-formulas can be placed in the modal symbol [ ] because only the situation of acquiring information on the quantum states is discussed.

We construct the deduction system which satisfies the soundness and completeness theorem with respect to the new models. *Sequent calculus* style of deduction system is constructed because it is compatible with OML [16] [17] [18].

As another approach to the problem, we consider another language for multiple particles.We want to treat the concepts as abstractly and simply as possible; therefore, we avoid symbols that primarily represent the concepts of quantum mechanics. However, there is a limit to what can be expressed with ordinary languages of OML itself. For instance, we can confirm that the following non implications of propositions of individual particles, which are important in multi particle systems, cannot be expressed. "Suppose that A is a proposition about the i-th space and B is a proposition about the j-th space. If $A \neq \top, A \neq \bot$, $B \neq \top$, and $B \neq \bot$, there exists states x, y such that $x \models A$, $x \not\models B$, $y \not\models A$, and $y \models B$". That is, in tensor space $H_3 \otimes H_3$ of 3D Hilbert space H_3, consider the following propositions. "The value of a physical quantity (which is associated with $|0>, |1>, |2>$) of the first particle is 0." "The value of a physical quantity (which is associated with $|0>, |1>, |2>$) of the second particle is 1." At $|0> \otimes |0> \in H_3 \otimes H_3$, the first proposition is true but the second is false; moreover, at $|1> \otimes |1>$, the first proposition is false but the second proposition is true.

Because of these circumstances, we extend a language that is not as complicated as possible.

Importantly, we use the relationship between OML and modal logic **B**. **B** denotes the logic developed on the frame assuming symmetry and reflectivity in the binary relation of the Kripke frame. **B** and OML are associated by McKinsey-Tarski transfer [9]. This correspondence is the same as that of intuitionistic logic and modal logic **S4**. As in the case of **S4** and intuitionistic logic, the corresponding modal logic can express a finer concept via a formula. For example, in both OML and intuitionistic logic, negation can be decomposed into $\Box\neg$ in modal logic, and $\Box$ and $\neg$(classical negation) can be separately used. Because it is convenient to handle $\Box$ alone, we develop the language and models based on **B** rather than OML.

A quantification symbol $\forall p_i$ is used for propositional variables. That is, $\forall p_i(\phi)$ is added to the definition of formulas. This is necessary to express properties such as entanglement concisely. This conceptually belongs to the category of the *second-order* propositional logic; however, only the quantification of propositional variables is employed, and not the quantification of the entire formula. Therefore, intuitively, the complex problems in the second-order propositional logic do not occur and can be handled fairly simply.

Using this language has the disadvantage of being a bit more complicated than the previous language, but it has the advantage of increasing the expressiveness of the model's conditions. In this study, the correspondences between various formulas with the above new definition and model conditions for a Hilbert space are proven. Some examples are shown below.($A_i, B_i, \ldots$ represent formulas which includes only $p_i, q_i, r_i, \ldots$as propositional variables).

For all $x, y \in W$, there exists $z \in W$ such that xRz and zRy.

$$\Box\Box A \to \Box^n A \quad \text{(for each } n \in \mathbb{N})$$

Each propositional variable represents a one-dimensional subspace of each Hilbert space.

$$(p_i \wedge A_i) \to \Box\Box(p_i \to A_i)$$

Non-implications of propositions of an individual particle.

$$(\neg\Box\Box A_i \wedge \neg\Box\Box\neg A_i \wedge \neg\Box\Box B_j \wedge \neg\Box\Box\neg B_j) \to \Diamond\Diamond(\neg A_i \wedge B_j) \wedge \Diamond\Diamond(A_i \wedge \neg B_j) \quad (i \neq j)$$

"Particle i and j are entangled"

$$\mathcal{E}_{i,j} = \forall p_i(\neg p_i) \wedge \forall q_j(\neg q_j) \wedge \forall p_i \exists q_j [p_i] q_j \wedge \forall q_j \exists p_i [q_l] p_i$$

The contributions of this study are the following.

1. New abstract logical frameworks and models for dealing with propositions about multiple agents and quantum particles are proposed.

2. A deduction system for new models that holds soundness and completeness is constructed.
3. We show that important conditions on models can be expressed with a little development of the language, and prove that these formulas are valid if a model satisfies specific conditions.

References

[1] Aerts, D.: Description of compound physical systems and logical interaction of physical systems. E. G. Beltrametti., B. C. van Fraassen (eds.), Current Issues on Quantum Logic, Ettore Majorana, International Science Series, Physical Sciences, (8), 381-–405 (1981)

[2] Baltag, A., Smets, S.: The logic of quantum programs. QPL 2004. 39–56 (2004)

[3] Baltag, A., Smets, S.: Correlated Knowledge: an Epistemic-Logic View on Quantum Entanglement. International Journal of Theoretical Physics. 49(12), 3005–3021 (2010)

[4] Baltag, A., Smets, S.: A Dynamic-Logical Perspective on Quantum Behavior. Studia Logica. 89, 187-–211 (2008)

[5] Beltrametti, E., Dalla Chiara M. L., Giuntini, R, Leporini, R., Sergioli, G.: A Quantum Computational Semantics for Epistemic Logical Operators. Part I: Epistemic Structures. International Journal of Theoretical Physics. 53(10), 3279–3292 (2013)

[6] Beltrametti, E., Dalla Chiara M. L., Giuntini, R, Leporini, R., Sergioli, G.: A Quantum Computational Semantics for Epistemic Logical Operators. Part II: Semantics. International Journal of Theoretical Physics. 53(10), 3293–3307 (2013)

[7] Birkhoff, G., Von Neumann, J.: The Logic of Quantum Mechanics. The Annals of Mathematics. 37(4), 823–843 (1936)

[8] Cattaneo, G., Dalla Chiara M. L., Giuntini, R., Paoli F: Quantum Logic and Nonclassical Logics. Handbook of Quantum Logic and Quantum Structures: Quantum Structures. Kurt Engesser, Dov M. Gabbay, Daniel Lehmann (eds.), Elsevier. (2007)

[9] Chiara, M. L. D., Giuntini, R.: Quantum Logics. Gabbay, D. M., Guenthner, F. (ed.): Handbook Of Philosophical Logic 2nd Edition. 6 (1), 129–228 (2002)

[10] van Ditmarsch, H. P., van der Hoek, W., Kooi, B.: Dynamic Epistemic Logic. Springer, Berlin. (2007)

[11] Kalmbach, G.: Orthomodular Logic. MLQ Math. Log. Q. 20, 295–406 (1974)

[12] Kawano, T.: Advanced Kripke Frame for Quantum Logic. Proceedings of 25th Workshop on Logic, Language, Information and Computation. 237–249 (2018)

[13] Kawano, T.: Studies on Implications and Sequent Calculi for Quantum Logic. Tokyo Institute of technology, Ph.D. thesis. (2018)

[14] Negri, S.: Proof Analysis in Modal Logic. Journal of Philosophical Logic. 34, 507–544 (2005)

[15] Negri, S.: Proof theory for modal logic. Philosophy Compass. 6(8), 523–538 (2011)

[16] Nishimura, H.: Sequential Method in Quantum Logic. The Journal of Symbolic Logic. 45(2), 339–352 (1980)

[17] Nishimura, H.: Proof Theory for Minimal Quantum Logic I. International Journal of Theoretical Physics. 33(1), 103–113 (1994)

[18] Nishimura, H.: Proof Theory for Minimal Quantum Logic II. International Journal of Theoretical Physics. 33(7), 1427–1443 (1994)