Modal Nelson lattices and their associated twist structures

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This work is about one of the most challenging trends of research in non-classical logic which is the attempt to combine different non-classical approaches together, in our case many-valued and modal logic. This kind of combination offers the skill of dealing with modal notions like belief, knowledge, and obligations, in interaction with other aspects of reasoning that can be best handled using many-valued logics, for instance, vagueness, incompleteness, and uncertainty. In fact, the study that we are going to introduce could be especially interesting from the point of view of Theoretical Computer Science and Artificial Intelligence.
One of the best-known logical systems proposed for handling uncertainty is perhaps Possibilistic logic [3, 4] which is able to reason with graded (epistemic) beliefs on classical propositions by means of necessity and possibility measures. Many authors have proposed generalizations for many-valued propositions [7, 5] but most of these settings have a limited scope since either only apply over finite truth-values (some times expanded with truth-constants and the Monteiro-Baaz's Δ operator) or only consider a language with finitely many variables or where the logic is defined over a two-tiered language, i.e. a flat modal language. Here, we are going to consider full modal logics defined over a *Nilpotent Minimum algebra* which allows interpreting conjunction in terms of min and negation in an involutive way.
In fact, by attempting to be as broad as possible, we introduce a more general approach based on modal Nelson lattices. Later, we show that modal Nilpotent algebras are a subvariety of them.

In order to reach our goal, we will first introduce an extension for modal setting of the one well-known construction of Nelson lattices called twist structures, whose importance has been growing in recent years within the study of algebras related to non-classical logics (see [1, 6, 9]). Our proposed extension is more general than others considered in the literature because it is not required to be monotone with respect to modal operators (see [8]).

We assume the reader know the main properties and definitions about residuated lattices and Heyting algebras. In addition, a residuated lattice is called involutive if it is bounded and it satisfies the double negation equation:

$$a = \neg\neg a.$$

A Nelson residuated lattice or simply Nelson lattice (N3) is an involutive residuated lattice satisfying:

$$((a^2 \to b) \wedge ((\neg b)^2 \to \neg a)) \to (a \to b) = \top.$$

Definition 1. Given a Heyting algebra $\mathbf{A}$, we shall denote by $D(\mathbf{A})$ the filter of dense elements of $\mathbf{A}$, i.e. $D(\mathbf{A}) = \{a \in A : \neg a = \bot\}$.

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A filter F of $\mathbf{A}$ is said to be Boolean provided the quotient $\mathbf{A}/F$ is a Boolean algebra. It is well known and easy to check that a filter F of the Heyting algebra $\mathbf{A}$ is Boolean if and only if $D(\mathbf{A}) \subseteq F$. The Boolean filters of $\mathbf{A}$, ordered by inclusion, form a lattice, having the improper filter A as the greatest element and $D(\mathbf{A})$ as the smallest element.
With all these elements, we can reproduce the twist-structures corresponding to N3-lattices.

Theorem 2. *(Sendlewski + Theorem 3.1 in [1].) Given a Heyting algebra*

$$\mathbf{H} = \langle H, \wedge, \vee, \to, \top, \bot \rangle$$

and a Boolean filter F of $\mathbf{H}$ let

$$R(\mathbf{H}, F) := \{(x, y) \in H \times H : x \wedge y = \bot \text{ and } x \vee y \in F\}.$$

Then we have:

1. $\mathbf{R}(\mathbf{H}, F) = \langle R(\mathbf{H}, F), \wedge, \vee, *, \Rightarrow, \bot, \top \rangle$ *is a Nelson lattice, when the operations are defined as follows:*
   - $(x, y) \vee (s, t) = (x \vee s, y \wedge t)$,
   - $(x, y) \wedge (s, t) = (x \wedge s, y \vee t)$,
   - $(x, y) * (s, t) = (x \wedge s, (x \to t) \wedge (s \to y))$,
   - $(x, y) \Rightarrow (s, t) = ((x \to s) \wedge (t \to y), x \wedge t)$,
   - $\top = (\top, \bot)$, $\bot = (\bot, \top)$.
2. $\neg(x, y) = (y, x)$,
3. *Given a Nelson lattice $\mathbf{A}$, there is a Heyting algebra $\mathbf{H}_{\mathbf{A}}$, unique up to isomorphisms, and a unique Boolean filter $F_{\mathbf{A}}$ of $\mathbf{H}_A$ such that $\mathbf{A}$ is isomorphic to $\mathbf{R}(\mathbf{H}_{\mathbf{A}}, F_{\mathbf{A}})$.*

Remark 3. Let $\mathbf{A}$ be a Nelson lattice. Let us consider $H = \{a^2 : a \in \mathbf{A}\}$ with the operations $a \star^* b = (a \star b)^2$ for every binary operation $\star \in \mathbf{A}$. Then,

$$\mathbf{H}^* = \langle H, \vee^*, \wedge^*, \to^*, 0, 1 \rangle$$

is a Heyting algebra ([10]).

Now, for our aim, we need to introduce some definitions of modal algebras.

Definition 4. A modal Heyting algebra $M\mathbf{A}$ is an algebra $\langle \mathbf{A}, \Box, \Diamond \rangle$ such that the reduct $\mathbf{A}$ is an Heyting algebra, $\Box$ and $\Diamond$ are two binary operators and, for all $a, b \in A$,

$$\text{if } a \wedge b = \bot \text{ then } \Box a \wedge \Diamond b = \bot. \tag{1}$$

Modal Heyting algebras obviously form a quasivariety and, at the present, we do not know whether this class is in fact a variety or not. However, there is well known extension of this quasi-variety that is a variety called *normal* modal Heyting algebra. It is obtaned by including the following equations:

3. $\neg \Diamond a = \Box \neg a$,
4. $\Box(a \to b) \to (\Box a \to \Box b) = \top$,

5. $\Box\top = \top$.

Note that (1) implies that $\Box a \wedge \Diamond\neg a = \bot$ and $\Box\neg a \wedge \Diamond a = \bot$, therefore, we can conclude $\Diamond\neg a \leq \neg\Box a$ and $\Box\neg a \leq \neg\Diamond a$. In addition, if (5) is assumed, we have $\Diamond\bot = \bot$.

Definition 5. A modal N3-lattice (for short MN3-lattice) is an algebra $\langle \mathbf{A}, \blacksquare, \blacklozenge\rangle$ such that the reduct $\mathbf{A}$ is an N3-lattice and, for all $a, b \in A$,

1. $\blacklozenge a = \neg\blacksquare\neg a$,
2. if $a^2 = b^2$ then $(\blacksquare a)^2 = (\blacksquare b)^2$ and $(\blacklozenge a)^2 = (\blacklozenge b)^2$,
3. if $(a \wedge b)^2 = \bot$ then $(\blacksquare a \wedge \blacklozenge b)^2 = \bot$.

In addition, $\mathbf{A}$ is said to be regular if it satifies the following:

4. $\blacksquare(a \wedge b) = \blacksquare a \wedge \blacksquare b$.

Moreover, if $\mathbf{A}$ is a regular modal N3-lattice (for short RMN3-lattice) by using (1) and (4), we can conclude:

4′. $\blacklozenge(a \vee b) = \blacklozenge a \vee \blacklozenge b$.

Finally, we say that a modal Nelson lattice is normal if it is regular and, in addition, satisfies:

5. $\blacksquare\top = \top$.

In this case, we can reproduce the following classical result on RMN3-lattices:

Lemma 6. *If* $\mathbf{N}$ *is a regular modal N3-lattice then it satisfies the next monotony properties:*

$$\text{if } a^2 \leq b \text{ then } (\blacksquare a)^2 \leq \blacksquare b, \quad \text{and} \quad \text{if } (\neg a)^2 \leq \neg b \text{ then } (\neg\blacksquare a)^2 \leq \neg\blacksquare b.$$

Now we are ready to formulate the first result of this work.

Theorem 7. *Let* $\mathbf{H}$ *and* F *be a modal Heyting algebra as defined in 4 and a Boolean filter satisfying:*

$$\text{if } a \wedge b = \bot \text{ and } a \vee b \in F \text{ then } \Box a \vee \Diamond b \in F.$$

Then, $\mathbf{R}(\mathbf{H}, F) = \langle R(\mathbf{H}, F), \wedge, \vee, *, \Rightarrow, \bot, \top, \blacksquare, \blacklozenge\rangle$ *is a Modal Nelson lattice, where the operators* $\blacksquare, \blacklozenge$ *are defined as follows:*

$$\blacksquare(x, y) = (\Box x, \Diamond y), \quad \text{and} \quad \blacklozenge(x, y) = (\Diamond x, \Box y).$$

Now, we are going to extend the representation of Nelson lattice in terms of Heyting algebras from Theorem 2 to the modal context. First we need introduce the next result.

Lemma 8. *Let* $\mathbf{N}$ *be a MN3-lattice. Consider* $\mathbf{H}^* = \langle H, \vee^*, \wedge^*, \to^*, \bot, \top, \Box^*, \Diamond^*\rangle$ *with* $H = \{a^2 : a \in N\}$ *and operators* $\vee^*, \wedge^*, \to^*$ *as in Remark 3 and modal operators as follows*

$$\Box^* a = (\blacksquare a)^2, \quad \text{and} \quad \Diamond^* a = (\blacklozenge a)^2$$

for every $a \in H$*. Then* $\mathbf{H}^*$ *is a modal Heyting algebra. In addition, if we take* $F = \{(a \vee \neg a)^2 : a \in N\}$*, then* F *is a Boolean filter satisfying*

$$\text{if } a \vee^* b \in F \text{ and } a \wedge^* b = \bot \text{ then } \Box^* a \vee^* \Diamond^* b \in F$$

for every $a, b \in H$*.*

A direct consequence of previous Lemma is our main result:

Theorem 9. *Let* $\mathbf{N}$ *be a modal N3-lattice. Then* $\mathbf{N}$ *is isomorphic to* $\mathbf{R}(\mathbf{H}^*, F)$ *as defined in Theorem 2 by taking* F *as in the previous lemma.*

Now, we would like to consider an important class of bounded residuated lattices which is the variety $\mathcal{MTL}$ determined by the prelinearity equation:

$$(a \to b) \vee (b \to a) = \top.$$

The involutive members of $\mathcal{MTL}$ satisfying the following equation are called nilpotent minimum algebras:

$$(a * b \to \bot) \vee (a \wedge b \to a * b) = \top.$$

In addition, it is well-known that every Nelson lattice satisfying prelinearity is a nilpotent minimum algebra (see [1, Theorem 6.16]). As usual, a Gödel algebra is a Heyting algebra that satisfies the prelinearity equation. Obviously, we can adapt Definition 4 for modal Gödel algebras and Definition 5 for modal nilpotent minimum algebra (MNM-algebras for short). Furthermore, it is easy to reproduce Theorem 7 giving a twist-construction of modal nilpotent minimum algebras in terms of modal Gödel algebras.
Obviously, MNM-algebras form a quasivariety. However, we are interesting in considering one of their subvarieties that we call pseudo-monadic nilpotent minimum algebras (see [2]). These algebras give us one of the possible algebraic semantics of Possibilistic logic.

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