Local Modal Product Logic is decidable

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Expansions with modal operators of many-valued logics have been proposed and studied in the literature following two main approaches. In this work, we contribute to the one introduced by Fitting [7, 8] and Hajek [9], which is based on considering a semantical definition of these logics which enriches the Kripke semantics with evaluations over corresponding many-valued algebras. In the literature, special attention has been devoted to modal expansion of the fuzzy logics associated with the basic continuous t-norms: Łukasiewicz modal logics [10], Gödel modal logics [3, 4, 12] and Product modal logics [15]. Most of these studies focus on the logics arising from the semantics with classical Kripke frames (namely, where the accessibility relation between worlds of the Kripke models is still a binary relation, and it is not valued over the algebra), and where the variables at the worlds of the model are the only elements evaluated over the corresponding algebras. In the following, we will refer by minimal modal fuzzy logics to those defined in this fashion, with $\Box$ ($\Box$-fragment), $\Diamond$ ($\Diamond$-fragment) or both modal operators (bi-modal logic). In general, the operators $\Box$ and $\Diamond$ are not interdefinable, and the logic with the two modal operators is possibly strictly weaker than the addition of the corresponding two mono-modal fragments [12]. Further, two logical consequences naturally arise from the same semantics: the local (where truth of the premises implies truth of the consequence world-wise) and global (where truth of the premises in the whole model implies the same for the consequence).

In different works, several of the decision problems concerning minimal modal fuzzy logics have been closed. Due to their very different characteristics, the studies in each case exploit particularities of each of the logics, relying little on general tools. It is known that the minimal local modal logics expanding Gödel logic are decidable [1, 2], that global modal Łukasiewicz[1] and bi-modal Product logics are undecidable [14] and that local modal Łukasiewicz logic is decidable [13]. Further, it is known that the analogous of the previous local logics for many-valued Kripke frames are decidable, which can be found in the same publications and, for the product case, it follows from the results in [5].

Two main questions concerning decidability of the previous minimal logics remain open: it is not known whether global Gödel modal logics are decidable or not, and it was also not known whether local bi-modal Product logic was decidable. Remarkably enough, the approach used in [3, 4] cannot be used to solve the first question, and the approach from [5] also does not serve as inspiration for proving decidability of local bi-modal Product logic for models with crisp accessibility relation.

In this work, we answer positively to the second open problem, and show that local bi-modal Product logic is decidable. Let us formally introduce the logic and sketch the ideas that allow us to conclude the decidability result stated before. Let $\mathcal{V}$ be a countable set of variables, and Fm the set of formulas over $\mathcal{V}$ with the algebraic language $\langle \odot/2, \rightarrow /2, \bot/0, \Box/1, \Diamond/1\rangle$. The interpretations of the symbols $\odot, \rightarrow$ and $\bot$ in a product algebra $\mathbf{A}$ is the natural one corresponding to its algebraic operations. For the definition of Product algebra and Product logic, see for instance [9].

[1] In which $\Box$ and $\Diamond$ are inter-definable, and so minimal fragments concerning modal operators all collapse.

Definition 1.1. Let $\mathbf{A}$ be a Product algebra. A (crisp) A-Kripke model $\mathfrak{M}$ is a structure $\langle W, R, e\rangle$ where W is a non-empty set, R is a binary relation over W and $e\colon W \times \mathcal{V} \to A$. A Kripke model uniquely determines an $\mathbf{A}$-Kripke model by extending the evaluation[2] to $e\colon W \times Fm \to A$ as follows:

$$e(v, \bot) \coloneqq 0^{\mathbf{A}}, \qquad e(v, \varphi \star \psi) \coloneqq e(v, \varphi) \star^{\mathbf{A}} e(v, \psi) \qquad \text{for } \star \in \{\odot, \to\}$$
$$e(v, \Box\varphi) \coloneqq \bigwedge_{Rvw} e(w, \varphi), \qquad e(v, \Diamond\varphi) \coloneqq \bigvee_{Rvw} e(w, \varphi)$$

We let $\mathbb{K}_\Pi$ denote the class of all $[0,1]_\Pi$-Kripke models, for $[0,1]_\Pi$ the standard Product algebra. For a set of formulas $\Gamma \cup \{\varphi\} \subseteq Fm$, we will write $\Gamma \vdash_{\mathbb{K}_\Pi} \varphi$ whenever for every $\mathfrak{M} \in \mathbb{K}_\Pi$ and every $v \in W$, if $e(v, \gamma) = 1$ for each $\gamma \in \Gamma$ then $e(v, \varphi) = 1$ as well. This entailment relation is what we referred to in the introduction as the local bi-modal Product logic. For convenience, for a model $\mathfrak{M}$ we write $\Gamma \not\vdash_{\mathfrak{M}} \varphi$ whenever there is some $v \in W$ for which $e(v, \Gamma) \subseteq \{1\}$ and $e(v, \varphi) < 1$.

The main result we will present in the conference is the following.

Theorem 2.1. *For a finite set of formulas $\Gamma \cup \{\varphi\} \subset_\omega Fm$, the problem of determining whether $\Gamma \vdash_{\mathbb{K}_\Pi} \varphi$ is decidable. Consequently, the set of theorems of the logic $\vdash_{\mathbb{K}_\Pi}$ is recursive.*

In the rest of the abstract, we will sketch the ideas that allow to prove the previous result.

A formula formula $\heartsuit\varphi$ starting with a modality $\heartsuit \in \{\Box, \Diamond\}$ is said to be witnessed in a world v of a model $\mathfrak{M}$ if there is a world w with Rvw and such that $e(v, \heartsuit\varphi) = e(w, \varphi)$. A model is witnessed if every formula is witnessed at every world of the model. It is known that modal product logic, as predicate product logic, is not complete with respect to witnessed models, which contrasts with the Łukasiewicz case. Nevertheless, it is complete with respect to so-called quasi-witnessed models. These are models where unwitnessing situations are rather limited: for each world v in the model and each formula $\heartsuit\varphi$ starting with a modality, either the formula is witnessed in v or the formula is of the form $\Box\varphi$ and $e(v, \Box\varphi) = 0$.

In this work, in order to prove decidability of $\vdash_{\mathbb{K}_\Pi}$, we rely in a more specific result, of which quasi-witnessed completeness is a corollary. In [11] it is proven that predicates (and so, modal) product logic is complete with respect to models valued over a particular product algebra.

Definition 2.2. The ***lexicographic sum*** $\mathbb{R}^{\mathbb{Q}} = \langle \mathbb{R}^{\mathbb{Q}}, +, \leqslant\rangle$ is the ordered abelian group of functions $f\colon \mathbb{Q} \to \mathbb{R}$ whose support is well-ordered (namely, such that $\{q \in \mathbb{Q}\colon f(q) \neq 0\}$ is a well-ordered subset of $\mathbb{Q}$). Addition is defined component-wise and the ordering on $\mathbb{R}^{\mathbb{Q}}$ is lexicographic.

The transformation $\mathfrak{B}$ introduced in [6] can be applied to the previous l-group, obtaining a product chain. Let us denote $(\mathbb{R}^{\mathbb{Q}})^- = \{a \in \mathbb{R}^{\mathbb{Q}}\colon a \leqslant \mathbf{0}\}$, where $\mathbf{0}$ stands for the neutral element of the group, namely, the constant function 0. Then, $\mathfrak{B}(\mathbb{R}^{\mathbb{Q}})$ is the product algebra with universe $(\mathbb{R}^{\mathbb{Q}})^- \cup \{\bot\}$ where the order is inherited from $\mathbb{R}^{\mathbb{Q}}$ (thus $\mathbf{0}$ is the maximum element) and $\bot$ is the minimum element, and the operations of the algebra are defined by letting

$$x \odot y = \begin{cases} x + y & \text{if } x, y \in (\mathbb{R}^{\mathbb{Q}})^-, \\ \bot & \text{otherwise} \end{cases} \qquad x \to y = \begin{cases} \mathbf{0} \wedge (y - x) & \text{if } x, y \in (\mathbb{R}^{\mathbb{Q}})^-, \\ \mathbf{0} & \text{if } x = \bot \\ \bot & \text{if } y = \bot \text{ and } x \in (\mathbb{R}^{\mathbb{Q}})^- \end{cases}$$

Theorem 2.9 from [11] is stated for predicate product logic (over all product chains), and so, we can restrict it naturally to the modal standard product logic as follows:

[2]If the corresponding infima/suprema do not exist, does values are undefined.

Corollary 2.3 (From Theorem 2.9, [11]). *Let $\Gamma \cup \{\varphi\}$ a set of modal formulas such that $\Gamma \not\vdash_{\mathbb{K}_\Pi} \varphi$. Then there is a countable, quasi-witnessed $\mathfrak{B}(\mathbb{R}^{\mathbb{Q}})$-Kripke structure $\mathfrak{M}$ such that $\Gamma \not\vdash_{\mathfrak{M}} \varphi$.*

Models over the above algebra are not only quasi-witnessed, but, when analyzed paying attention to a finite set of formulas Σ[3], they satisfy more specific conditions. For if we have a formula $\Box\varphi \in \Sigma$ unwitnessed in a world v in a model, it holds that there is some world $v_{\Box\varphi}$[4] with $Rvv_{\Box\varphi}$ and such that there is a negative rational number q for which

$$
\begin{aligned}
&e(v_{\Box\varphi}, \varphi)[q] <0,\\
&e(v_{\Box\varphi}, \varphi)[p] =0 \text{ for all } p < q, \text{ and}\\
&e(v_{\Box\varphi}, \psi)[p] =0 \text{ for all } p \leqslant q \text{ and all } \Box\psi \in \Sigma \text{ such that } e(v, \Box\psi) > 0.
\end{aligned}
$$

For each $\psi, \Box\varphi \in \Sigma$, $v \in W$ and $\Box\varphi$ unwitnessed in v, consider the element of the algebra $\alpha(v_{\Box\varphi}, \psi)$ defined by $\perp$ if $e(v_{\Box\varphi}, \psi) = \perp$ and, in other case,

$$\alpha(v_{\Box\varphi}, \psi)[p] := \begin{cases} 0 & \text{if } p > q \\ e(v_{\Box\varphi}, \psi)[p] & \text{otherwise} \end{cases}$$

Observe that $\perp < \alpha(v_{\Box\varphi}, \varphi) < \top$, and $\alpha(v_{\Box\varphi}, \psi) = \top$ for each $\Box\psi \in \Sigma$ such that $e(v, \Box\psi) > \perp$.

By iterating the previous idea, it is possible to extend any model with additional worlds in such a way that, for every unwitnessed formula $\Box\varphi$ at a world v, we have two special successors of v, $v_{\Box\varphi}$ and $v'_{\Box\varphi}$, such that for each formula $\psi \in \Sigma$ there is a value $\alpha_{\langle v, \Box\varphi\rangle}(\psi)$ for which:

$$
\begin{aligned}
\perp < \alpha_{\langle v, \Box\varphi\rangle}(\varphi) &< \top\\
\alpha_{\langle v, \Box\varphi\rangle}(\psi) &= \top \text{ for each } \Box\psi \in \Sigma \text{ with } e(v, \Box\psi) > \perp\\
\alpha_{\langle v, \Box\varphi\rangle}(\psi) &= \perp \text{ iff } e(v_{\Box\varphi}, \psi) = \perp,\\
\text{if } e(v_{\Box\varphi}, \psi_1) \leqslant e(v_{\Box\varphi}, \psi_2) &\text{ then } \alpha_{\langle v, \Box\varphi\rangle}(\psi_1) \leqslant \alpha_{\langle v, \Box\varphi\rangle}(\psi_2) \text{ for each } \psi_1, \psi_2 \in \Sigma,\\
e(v'_{\Box\varphi}, \psi) &= e(v_{\Box\varphi}, \psi) + \alpha_{\langle v, \Box\varphi\rangle}(\psi)
\end{aligned}
$$

Restricting this extended model $\mathfrak{M}^+$ to the worlds witnessing the witnessed formulas from Σ and to the pairs above whenever an unwitnessed formula appears, we obtain a finite model $\mathfrak{M}'$ with the above characteristics. Observe that the values themselves are not relevant, and only the information concerning which successor is the witness of each formula, and the above information for what concerns the unwitnessed formulas. All this information can be faithfully encoded with a derivation in the propositional product logic with Δ, let say $\Gamma_{\mathfrak{M}'} \vdash_\Pi \varphi_{\mathfrak{M}'}$, whose involved formulas are constructively defined from the original ones and the model. Since all the possible combinations of unwitnessed formulas form a finite set, the set of possible models from which we start can be also taken as finite, $\{\mathfrak{M}_1 \dots, \mathfrak{M}_n\}$ (where n is bounded exponentially by the modal depth of Σ), which allows to claim the following:

Proposition 3.1. *If $\Gamma \not\vdash_{\mathbb{K}_\Pi} \varphi$ there is some $\mathfrak{M}_i$ in $\{\mathfrak{M}_1 \dots, \mathfrak{M}_n\}$ such that $\Gamma_{\mathfrak{M}'_i} \not\vdash_{\Pi_\Delta} \varphi_{\mathfrak{M}'_i}$.*

The key part is that, from the previous propositional condition we can build back a $[0,1]_\Pi$-Kripke model (which will be infinite whenever any of the variables associated to the values $\alpha_{\langle v, \Box\varphi\rangle}(\psi)$ is different from 1) with the desired property, namely:

[3] Since we will be addressing the decidability question for entailments of a formula φ from a finite set of premises Γ, in practice Σ will be the closure under subformulas of the set $\Gamma \cup \{\varphi\}$.

[4] In fact, infinitely many ones.

Proposition 3.2. *If there is some $\mathfrak{M}_i$ in $\{\mathfrak{M}_1 \ldots, \mathfrak{M}_n\}$ such that $\Gamma_{\mathfrak{M}'_i} \not\vdash_{\Pi_\Delta} \varphi_{\mathfrak{M}'_i}$, then we can construct an $[0,1]_\Pi$-Kripke model $\mathfrak{N}$ from $\Gamma_{\mathfrak{M}'_i} \cup \{\varphi'_{\mathfrak{M}_i}\}$ such that $\Gamma \not\vdash_{\mathfrak{N}_i} \varphi$.*

This construction is done by defining, for each variable associated to a value $\alpha_{\langle v, \Box\varphi\rangle}(\psi)$ that is below 1, an infinite set of points v_i in each of which the value of each formula $\psi \in \Sigma$ is sent to the value taken by ψ in v [5] multiplied by $\alpha_{\langle v, \Box\varphi\rangle}(\psi)^i$. This allows us to replicate, in a regular way, the behavior of the unwitnessed formulas, without affecting the others.

Since the logic $\vdash_{\Pi_\Delta}$ is decidable, and the previous constructions are recursive, this allows to immediately conclude the decidability of $\vdash_{\mathbb{K}_\Pi}$.

Furthermore, this approach proves that the modal logic arising from the Kripke models evaluated over all Product algebras and that arising from the models evaluated over the standard Product algebra coincide.

Acknowledgments
This project has received funding from the following sources: the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No. 101027914.

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[5] This comment refers to the fact that in $\Gamma_{\mathfrak{M}'_i} \cup \{\varphi'_{\mathfrak{M}_i}\}$, the propositional variables are of the form x_v, for x in $\mathcal{V}$ or a formula begining by a modality. Thus, we can talk about the value of a formula in a world even if we are in fact working with a propositional homomorphism.

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