Degrees of the finite model property: The antidichotomy theorem

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The most common semantics for modal and superintuitionistic logics is Kripke semantics, which since its inception (in late 1950s and early 1960s), has become one of the main tools in the study of these logics. Logics that are sound and complete with respect to a class of Kripke frames are called *Kripke complete*. A solid body of completeness results for Kripke semantics has been obtained culminating in Sahlqvist canonicity and correspondence results establishing Kripke completeness for a large class of modal logics (see, e.g., [2]; for Sahlqvist theory for superintuitionistic logics see [13]). However, examples of Kripke incomplete logics began to emerge in the 1970s (see, e.g., [6, Ch. 6]).

In order to shed light on the phenomenon of Kripke completeness, Fine [10] associated with each normal modal logic L a cardinal that measures the degree of incompleteness of L. More precisely, let $\mathsf{Fr}(\mathsf{L})$ be the class of Kripke frames validating L. We say that the *degree of incompleteness* of L is the cardinal κ if there are exactly κ normal modal logics L' such that $\mathsf{Fr}(\mathsf{L}') = \mathsf{Fr}(\mathsf{L})$. Notice that all but one of these L' are Kripke incomplete.

Blok [4, 5] gave a very unexpected characterization of degrees of incompleteness, which became known as *Blok's dichotomy theorem*. It states that a normal modal logic L has the degree of incompleteness either 1 or $2^{\aleph_0}$; it is 1 iff L is a join-splitting logic; otherwise it is $2^{\aleph_0}$. We refer to [18] and [16] for a detailed discussion of Blok's dichotomy and its importance in modal logic.

Blok's result implies that some of the most studied normal modal logics, such as $\mathsf{K4}$ (the logic of transitive Kripke frames) and $\mathsf{S4}$ (the logic of reflexive and transitive Kripke frames), have the degree of incompleteness $2^{\aleph_0}$. However, the logics sharing the frames with $\mathsf{K4}$ and $\mathsf{S4}$ are not necessarily normal extensions of $\mathsf{K4}$ or $\mathsf{S4}$. Thus, Blok's result does not automatically transfer to normal extensions of $\mathsf{K4}$ or $\mathsf{S4}$ (or, more generally, to normal extensions of a given normal modal logic L). There have been several attempts to investigate Blok's dichotomy for normal extensions of $\mathsf{K4}$ and $\mathsf{S4}$. However, this remains an outstanding open problem in modal logic [6, Prob. 10.5].

For a logic L, let $\mathsf{Fin}(\mathsf{L})$ be the class of finite Kripke frames validating L. We recall that L has the *finite model property* (*fmp* for short) if L is complete with respect to $\mathsf{Fin}(\mathsf{L})$. Clearly each logic with the fmp is Kripke complete. In addition, every finitely axiomatizable logic with the fmp is decidable by Harrop's theorem (see, e.g., [6, Thm. 16.13]).

Taking inspiration from degrees of incompleteness, it is natural to introduce a similar concept for the fmp. We say that the *degree of fmp* of a logic L is κ provided there exist exactly κ logics L' such that $\mathsf{Fin}(\mathsf{L}') = \mathsf{Fin}(\mathsf{L})$. As with the degree of incompleteness, all but one of such L' lack the fmp. Our main result establishes a complete opposite of Blok's dichotomy theorem for superintuitionistic logics and transitive (normal) modal logics. Namely, we prove that if κ is a nonzero cardinal such that $\kappa \leq \aleph_0$ or $\kappa = 2^{\aleph_0}$, then there exists a superintuitionistic logic (or a transitive modal logic) L whose degree of fmp is κ. Under the Continuum Hypothesis (CH) this implies that each nonzero $\kappa \leq 2^{\aleph_0}$ is realized as the degree of fmp of some superintuitionistic logic (or some transitive modal logic). For this reason, we refer to this result as the *antidichotomy theorem for degrees of fmp.*

In [16, p. 409] Litak asks "if there is any nontrivial completeness notion for which the Blok dichotomy does not hold." Our main result provides such a nontrivial notion for superintuitionistic logics and transitive modal logics. It also provides a solution of a variant of [6, Prob. 10.5] when the degree of incompleteness is replaced with the degree of fmp.

To give more context, we recall that *superintuitionistic logics* are (axiomatic) extensions of the intuitionistic propositional calculus IPC. They have been studied extensively in the literature (see, e.g., [6]). In particular, there is a close connection between superintuitionistic logics and normal extensions of S4. The *Gödel translation* embeds IPC into S4 fully and faithfully [17]. Thus, each superintuitionistic logic L is embedded into a normal extension of S4, called a *modal companion* of L [6, Sec. 9.6]. Each L has many modal companions, but remarkably each L possesses a largest one. By Esakia's theorem [7, 9], the largest modal companion of IPC is the well-known Grzegorczyk logic Grz. Consequently, the largest modal companion of each superintuitionistic logic is a normal extension of Grz, and there exists an isomorphism between the lattice of superintuitionistic logics and the lattice of normal extensions of Grz (the Blok-Esakia theorem) [3, 7].

It is a consequence of Blok's dichotomy theorem that the degree of fmp of a normal extension of the basic modal logic K remains 1 or $2^{\aleph_0}$. Thus, in the lattice of all normal modal logics the dichotomy holds also for the degrees of fmp.

We conclude by discussing how we establish our main results. We first prove the antidichotomy theorem for degrees of fmp of superintuitionistic logics. We heavily rely on Esakia duality for Heyting algebras [8], as well as on Fine's completeness theorem for logics of bounded width [11] and the theory of splittings [6, Sec. 10.5]. Our proof is broken into two parts, depending on whether $\kappa \leq \aleph_0$ or $\kappa = 2^{\aleph_0}$.

When $\kappa \leq \aleph_0$ we work with extensions of the superintuitionistic logic KG, which was introduced by Kuznetsov and Gerčiu [12, 15] and bears their name. The logic KG is the logic of sums of one-generated Heyting algebras, the combinatorics of which allows to construct extensions of KG that lack the fmp [15, 14, 1]. First, we use Fine's completeness theorem to prove that KG is a join-splitting logic over IPC (for a similar result see [14]). Then we develop a method, utilizing a technique of [1], that produces an extension L of KG whose degree of fmp is κ for every nonzero cardinal $\kappa \leq \aleph_0$.

To show that there exist superintuitionistic logics whose degree of fmp is $2^{\aleph_0}$ we work with superintuitionistic logics of finite width. Transitive modal logics of finite width were introduced by Fine [11] who showed that each transitive modal logic of finite width has the fmp. The concept was adapted to superintuitionistic logics by Sobolev [19]. For every positive integer n, let BW_n be the least superintuitionistic logic of width n. We prove that if $n > 2$, the degree of fmp of BW_n is $2^{\aleph_0}$. This is done by a careful analysis of the combinatorics of posets of bounded width.

Under CH our results show that for every nonzero cardinal $\kappa \leq 2^{\aleph_0}$ there exists a superintuitionistic logic L whose degree of fmp is κ, thus yielding the antidichotomy theorem for degrees of fmp of superintuitionistic logics.

Finally, we transfer our results to the setting of modal logics. Following the notation of [6], for a normal modal logic L, let Next L be the lattice of normal extensions of L. We first use the Blok-Esakia theorem to prove our antidichotomy theorem for Next Grz. Next we show that for each normal modal logic $\mathsf{L} \subseteq \mathsf{Grz}$ with the fmp, the antidichotomy theorem holds for Next L provided Grz is a join-splitting logic above L. Since Grz is a join-splitting logic above both S4 and K4 and these logics have the fmp, it follows that the antidichotomy theorem holds for both Next S4 and Next K4.

References

[1] G. Bezhanishvili, N. Bezhanishvili, and D. de Jongh. The Kuznetsov-Gerciu and Rieger-Nishimura logics: the boundaries of the finite model property. *Logic and Logical Philosophy*, 17:73–110, 2008.

[2] P. Blackburn, M. de Rijke, and Y. Venema. *Modal Logic.* Cambridge University Press, 2001.

[3] W. J. Blok. *Varieties of Interior Algebras.* PhD thesis, University of Amsterdam, 1976.

[4] W. J. Blok. On the degree of incompleteness of modal logics. *Bulletin of the Section of Logic*, 7:167–175, 1978.

[5] W. J. Blok. On the degree of incompleteness of modal logics and the covering relation in the lattice of modal logics. Technical Report 78–07, Department of Mathematics, University of Amsterdam, 1978.

[6] A. Chagrov and M. Zakharyaschev. *Modal Logic*, volume 35 of *Oxford Logic Guides.* Oxford University Press, 1997.

[7] L. Esakia. On modal "companions" of superintuitionistic logics. In *VII Soviet Symposium on Logic (Russian) (Kiev, 1976)*, pages 135–136. 1976.

[8] L. Esakia. *Heyting Algebras. Duality Theory.* Springer, English translation of the original 1985 book. 2019.

[9] L. L. Esakia. On the variety of Grzegorczyk algebras. In *Studies in nonclassical logics and set theory (Russian)*, pages 257–287. "Nauka", Moscow, 1979.

[10] K. Fine. An incomplete logic containing S4. *Theoria*, 40:23–29, 1974.

[11] K. Fine. Logics containing $K4$, Part I. *J. Symbolic Logic*, 34:31–42, 1974.

[12] V. Ja. Gerčiu and A. V. Kuznetsov. The finitely axiomatizable superintuitionistic logics. *Soviet Mathematics Doklady*, 11:1654–1658, 1970.

[13] S. Ghilardi and G. Meloni. Constructive canonicity in non-classical logics. *Ann. Pure Appl. Log.*, 86(1):1–32, 1997.

[14] M. Kracht. Splittings and the finite model property. *J. Symbolic Logic*, 58(1):139–157, 1993.

[15] A. V. Kuznetsov and V. Ja. Gerčiu. Superintuitionistic logics and finite approximability. *Soviet Mathematics Doklady*, 11:1614–1619, 1970.

[16] T. Litak. Stability of the Blok theorem. *Algebra Universalis*, 58(4):385–411, 2008.

[17] J. C. C. McKinsey and A. Tarski. Some theorems about the sentential calculi of Lewis and Heyting. *J. Symbolic Logic*, 13:1–15, 1948.

[18] W. Rautenberg, M. Zakharyaschev, and F. Wolter. Willem Blok and modal logic. *Studia Logica*, 83(1-3):15–30, 2006.

[19] S. K. Sobolev. On finite-dimensional superintuitionistic logics. *Izv. Akad. Nauk SSSR Ser. Mat.*, 41(5):963–986, 1977.