On equational completeness theorems

T. Moraschini

Department of Philosophy, University of Barcelona, Spain
tommaso.moraschini@ub.edu

Abstract

A logic is said to admit an equational completeness theorem when it can be interpreted into the equational consequence relative to a class of algebras. We characterize logics admitting an equational completeness theorem that have at least one tautology. As a consequence, a protoalgebraic logic admits an equational completeness theorem precisely when it has a matrix semantics validating a nontrivial equation. While the problem of determining whether a logic admits an equational completeness theorem is shown to be decidable both for logics presented by a finite set of finite matrices and for locally tabular logics presented by a finite Hilbert calculus, it becomes undecidable for arbitrary logics presented by finite Hilbert calculi.

1 Introduction

By a *logic* [11] we understand a consequence relation $\vdash$ on the set of formulas Fm (built up with a denumerable set of variables) of some algebraic language that, moreover, is *substitution invariant* in the sense that for every $\Gamma \cup \{\varphi\} \subseteq Fm$ and every substitution σ,

$$\text{if } \Gamma \vdash \varphi, \text{ then } \sigma[\Gamma] \vdash \sigma(\varphi).$$

A logic $\vdash$ admits an *equational (soundness and) completeness theorem* if there are a set of equations $\boldsymbol{\tau}(x)$ and a class of similar algebras K such that for every set of formulas $\Gamma \cup \{\varphi\}$,

$$\begin{aligned}\Gamma \vdash \varphi \iff & \text{ for every } \boldsymbol{A} \in \mathsf{K} \text{ and } \vec{a} \in A,\\ & \text{if } \boldsymbol{A} \vDash \boldsymbol{\tau}(\gamma^{\boldsymbol{A}}(\vec{a})) \text{ for every } \gamma \in \Gamma, \text{ then } \boldsymbol{A} \vDash \boldsymbol{\tau}(\varphi^{\boldsymbol{A}}(\vec{a})).\end{aligned}$$

In this case, K is said to be an *algebraic semantics* for $\vdash$ (or a $\boldsymbol{\tau}$*-algebraic semantics* for $\vdash$). Accordingly, a logic admits an equational completeness theorem precisely when it has an algebras semantics.

For instance, the well-known equational completeness theorem for the classical propositional calculus **CPC** states that for every set of formulas $\Gamma \cup \{\varphi\}$,

$$\begin{aligned}\Gamma \vdash_{\mathbf{CPC}} \varphi \iff & \text{ for every Boolean algebra } \boldsymbol{A} \text{ and } \vec{a} \in A,\\ & \text{if } \boldsymbol{A} \vDash \gamma^{\boldsymbol{A}}(\vec{a}) \approx 1 \text{ for every } \gamma \in \Gamma, \text{ then } \boldsymbol{A} \vDash \varphi^{\boldsymbol{A}}(\vec{a}) \approx 1.\end{aligned}$$

Thus, the variety of Boolean algebras is an algebraic semantic for **CPC**.

The notion of an algebraic semantics was introduced by Blok and Pigozzi in the study of *algebraizable logics* [5], i.e., logics that are *equivalent* to equational consequences in the sense of [1, 2]. From this point of view, a logic has a $\boldsymbol{\tau}$-algebraic semantics K when it satisfies one half of this equivalence, namely it can be interpreted into the equational consequence relative to K by translating formulas into equations by means of the set of equations $\boldsymbol{\tau}(x)$.

Intrinsic characterizations of logics with an algebraic semantics have proved elusive, partly because equational completeness theorems can take unexpected forms. For instance, in view of Glivenko's theorem [12], for every set of formulas $\Gamma \cup \{\varphi\}$ of **CPC**,

$$\Gamma \vdash_{\mathbf{CPC}} \varphi \Longleftrightarrow \{\neg\neg\gamma \colon \gamma \in \Gamma\} \vdash_{\mathbf{IPC}} \neg\neg\varphi,$$

where **IPC** stands for the intuitionistic propositional calculus. Since Heyting algebras form an $\{x \approx 1\}$-algebraic semantics for **IPC**, one obtains

$$\begin{aligned}\Gamma \vdash_{\mathbf{CPC}} \varphi \Longleftrightarrow & \text{ for every Heyting algebra } \boldsymbol{A} \text{ and } \vec{a} \in A, \\ & \text{if } \boldsymbol{A} \vDash \neg\neg\gamma^{\boldsymbol{A}}(\vec{a}) \approx 1 \text{ for every } \gamma \in \Gamma, \text{ then } \boldsymbol{A} \vDash \neg\neg\varphi^{\boldsymbol{A}}(\vec{a}) \approx 1.\end{aligned}$$

Consequently, the variety of Heyting algebras is also an algebraic semantics for **CPC**, although certainly not the intended one [7, Prop. 2.6].

The fragility of the property of having an algebraic semantics was confirmed by Blok and Rebagliato, who showed that every logic possessing an idempotent connective admits an algebraic semantics [7, Thms. 3.1]. On the other hand, the existence of logics that do not possess any algebraic semantics is known since [3]. It is therefore sensible to wonder whether an intelligible characterization of logics with an algebraic semantics could possibly be obtained [16]. In this talk we provide a positive answer to this question for a wide family of logics.

2 Main results

We shall describe large families of logics with an algebraic semantics. To this end, it is convenient to isolate some limits cases: a logic is said to be *graph-based* when its language comprises only constant symbols and, possibly, a single unary connective. Needless to say, most interesting logics in the literature are *not* graph-based.

To tackle the case of logics that are not graph-based, we first introduce a general method for constructing algebraic semantics based on a universal algebraic trick known as *Maltsev's Lemma*, which provides a description of congruence generation in arbitrary algebras. More precisely, we establish the following, where $Var(\varphi)$ denotes the set of variables occurring in the formula φ.

Theorem 1. *Let $\vdash$ be a logic that is not graph-based. If $\vdash$ has a matrix semantics validating a nontrivial equation $\varphi \approx \psi$ such that $Var(\varphi) \cup Var(\psi) = \{x\}$, then $\vdash$ has an algebraic semantics.*

A logic $\vdash$ is said to be *locally tabular* if it has a matrix semantics whose algebraic reducts generate a locally finite variety.

Corollary 2. *If a logic is locally tabular and not graph-based, then it has an algebraic semantics.*

Another application of Theorem 1 consists in a description of logics with theorems possessing an algebraic semantics. Recall that a formula φ is said to be a *theorem* of a logic $\vdash$ when $\emptyset \vdash \varphi$. Furthermore, a logic $\vdash$ is called *assertional* [15] when it has a matrix semantics M for which there is a unary formula $\gamma(x)$ such that for every $\langle \boldsymbol{A}, F \rangle \in \mathsf{M}$, the term-function $\gamma^{\boldsymbol{A}} \colon A \to A$ is a constant function and its unique value a is such that $F = \{a\}$. Intermediate logics, as well as global consequences [13] of normal modal logics, are known to be assertional.

Theorem 3. *Let $\vdash$ be a nontrivial logic with a theorem φ such that $Var(\varphi) \neq \emptyset$. Then $\vdash$ has an algebraic semantics if and only if either $\vdash$ is assertional and graph-based or it is not graph-based and has a matrix semantics validating a nontrivial equation $\epsilon \approx \delta$ such that $Var(\epsilon) \cup Var(\delta) = \{x\}$.*

A logic $\vdash$ is said to be *protoalgebraic* [4, 8, 9, 10] if there exists a set of formulas $\Delta(x, y)$ such that $\emptyset \vdash \Delta(x, x)$ and $x, \Delta(x, y) \vdash y$. Nontrivial protoalgebraic logics are not graph-based and possess at least a theorem φ such that $Var(\varphi) \neq \emptyset$. This makes them amenable to the above theorem which, moreover, can be improved as follows:

Corollary 4. *A nontrivial protoalgebraic logic has an algebraic semantics if and only if it has a matrix semantics validating a nontrivial equation.*

In view of the above result, almost all reasonable protoalgebraic logics have an algebraic semantics. It is therefore natural to wonder whether they have also a *natural* algebraic semantics. There is, however, evidence against this conjecture, since, while the local consequence [13] of the normal modal logic **K** (resp. **K4** and **S4**) has an ad hoc algebraic semantics in view of the above corollary, it does not possess one based on the variety of modal algebras (resp. K4-algebras and interior algebras).

We conclude our journey among equational completeness theorems with some computational observations:

Theorem 5. *The following holds:*

(i) *The problem of determining whether logics presented by a finite set of finite matrices in a finite language have an algebraic semantics is decidable;*

(ii) *The problem of determining whether locally tabular logics presented by a finite set of finite rules in a finite language have an algebraic semantics is decidable;*

(iii) *The problem of determining whether logics presented by a finite set of finite rules in a finite language have an algebraic semantics is undecidable.*

The last item of the above result is established by means of a reduction to the classical halting problem for Turing machines [17]. This talk is based on the paper [14].

References

[1] W. J. Blok and B. Jónsson. Algebraic structures for logic. A course given at the 23rd Holiday Mathematics Symposium, New Mexico State University, 1999.

[2] W. J. Blok and B. Jónsson. Equivalence of consequence operations. *Studia Logica*, 83(1–3):91–110, 2006.

[3] W. J. Blok and P. Köhler. Algebraic semantics for quasi-classical modal logics. *The Journal of Symbolic Logic*, 48:941–964, 1983.

[4] W. J. Blok and D. Pigozzi. Protoalgebraic logics. *Studia Logica*, 45:337–369, 1986.

[5] W. J. Blok and D. Pigozzi. *Algebraizable logics*, volume 396 of *Mem. Amer. Math. Soc.* A.M.S., Providence, January 1989.

[6] W. J. Blok and D. Pigozzi. Algebraic semantics for universal Horn logic without equality. In A. Romanowska and J. D. H. Smith, editors, *Universal Algebra and Quasigroup Theory*, pages 1–56. Heldermann, Berlin, 1992.

[7] W. J. Blok and J. Rebagliato. Algebraic semantics for deductive systems. *Studia Logica, Special Issue on Abstract Algebraic Logic, Part II*, 74(5):153–180, 2003.

[8] J. Czelakowski. Algebraic aspects of deduction theorems. *Studia Logica*, 44:369–387, 1985.

[9] J. Czelakowski. Local deductions theorems. *Studia Logica*, 45:377–391, 1986.

[10] J. Czelakowski. *Protoalgebraic logics*, volume 10 of *Trends in Logic—Studia Logica Library*. Kluwer Academic Publishers, Dordrecht, 2001.

[11] J. M. Font. *Abstract Algebraic Logic - An Introductory Textbook*, volume 60 of *Studies in Logic - Mathematical Logic and Foundations*. College Publications, London, 2016.

[12] V. I. Glivenko. Sur quelques points de la logique de M. Brouwer. *Academie Royal de Belgique Bulletin*, 15:183–188, 1929.

[13] M. Kracht. *Modal consequence relations*, volume 3, chapter 8 of the Handbook of Modal Logic. Elsevier Science Inc., New York, NY, USA, 2006.

[14] T. Moraschini. On equational completeness theorems. Published online in the *Journal of Symbolic Logic*, 2021.

[15] D. Pigozzi. Fregean algebraic logic. In H. Andréka, J. D. Monk, and I. Németi, editors, *Algebraic Logic*, volume 54 of *Colloq. Math. Soc. János Bolyai*, pages 473–502. North-Holland, Amsterdam, 1991.

[16] J. G. Raftery. A perspective on the algebra of logic. *Quaestiones Mathematicae*, 34:275–325, 2011.

[17] A. M. Turing. On computable numbers, with and application to the Entscheidungsproblem. *Proceedings of the London Mathematical Society*, 42(2):230–265, 1936–1937. A correction 43 (1937), 544–546.