# Towards a non-integral variant of Łukasiewicz logic

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Łukasiewicz logic is one of the most prominent non-classical logics with very rich metamathematics and with deep connections with many areas of mathematics such as lattice-ordered Abelian groups, continuous model theory, rational polyhedra, Chang MV-algebras, algebraic probability theory, etc. [2,8,10]. One of its defining features is known algebraically as *integrality* (the maximal truth value is the unit of strong conjunction) or proof-theoretically as *weakening* (one can derive $\varphi \to \psi$ from $\psi$). There are numerous reasons to omit this condition and many of the resulting logics have been studied in the literature under the guise of substructural logics [6]. However none of them can, as of now, boast as deep connections to other areas of logic and mathematics as Łukasiewicz logic.

The existing approaches are arguably either too weak (e.g. the logic of GMV-algebras [7], which drops also commutativity of fusion and semilinearity), or too strong (e.g. Abelian logic [1, 9] which is contraclassical, i.e. proves claims such as $((\varphi \to \psi) \to \psi) \to \varphi$).

In the recent paper [3], motivated by issues of reasoning with graded predicates, a new logic was proposed based on particular residuated lattice $\boldsymbol{C}$ over the set $\overline{\mathbb{R}}$ of *all* real numbers extended with two limit points $+\infty$ and $-\infty$, in which

- the values in the open interval $(0, 1)$ are still intended as intermediate degrees of truth,
- 1 is intended as the least degree for definitively true statements and will interpret the truth-constant $\mathtt{t}$,
- values above 1 are also intended for definitively true statements and $+\infty$, the largest of these values, will interpret the truth-constant $\top$,
- 0 is intended as the largest degree for definitively false statements and will interpret the truth constant $\mathtt{f}$ (and thus be used to define the negation connective),
- values below 0 are also intended for definitively false statements and $-\infty$, the least of these values, will interpret the truth-constant $\bot$,
- the interpretation of connectives $\to$ and $\&$ are defined using the following table; note that for $x, y \in \mathbb{R}$ it is just the untruncated form of the interpretation of these connectives in Łukasiewicz logic:

| $x \,\&^{\boldsymbol{C}}\, y$ | $y = -\infty$ | $y \in \mathbb{R}$ | $y = +\infty$ |
|---|---|---|---|
| $x = -\infty$ | $-\infty$ | $-\infty$ | $-\infty$ |
| $x \in \mathbb{R}$ | $-\infty$ | $x + y - 1$ | $+\infty$ |
| $x = +\infty$ | $-\infty$ | $+\infty$ | $+\infty$ |

| $x \to^{\boldsymbol{C}} y$ | $y = -\infty$ | $y \in \mathbb{R}$ | $y = +\infty$ |
|---|---|---|---|
| $x = -\infty$ | $+\infty$ | $+\infty$ | $+\infty$ |
| $x \in \mathbb{R}$ | $-\infty$ | $1 - x + y$ | $+\infty$ |
| $x = +\infty$ | $-\infty$ | $-\infty$ | $+\infty$ |

The algebra $\boldsymbol{C} = \langle \overline{\mathbb{R}}, \wedge^{\boldsymbol{C}}, \vee^{\boldsymbol{C}}, \&^{\boldsymbol{C}}, \to^{\boldsymbol{C}}, \mathtt{f}^{\boldsymbol{C}}, \mathtt{t}^{\boldsymbol{C}}, \bot^{\boldsymbol{C}}, \top^{\boldsymbol{C}} \rangle$ is a IUL-*chain* (cf. [5]) and the negation $\neg^{\boldsymbol{C}} x = x \to^{\boldsymbol{C}} \mathtt{f}^{\boldsymbol{C}}$ is the involutive function

$$\neg^{\boldsymbol{C}} x = \begin{cases} 1 - x & \text{for } x \in \mathbb{R} \\ -\infty & \text{for } x = +\infty \\ +\infty & \text{for } x = -\infty \end{cases}$$

The chain $\boldsymbol{C}$ is related to a particular family of standard IUL-chains, denoted as $\mathcal{A}(\circ_{CR}, f)$, which are given by the cross ratio uninorm $\circ_{CR}$:

$$a \circ_{CR} b = \begin{cases} \dfrac{ab}{ab + (1-a)(1-b)}, & \text{if } \{a, b\} \neq \{0, 1\}, \\ 0, & \text{otherwise,} \end{cases}$$

and its residuum $\Rightarrow_{CR}$, and by fixing the interpretation of $\mathtt{f}$ as $f$ (clearly $\bot$, $\mathtt{t}$, and $\top$ are interpreted as 0, $\frac{1}{2}$, and 1). We show that $\boldsymbol{C}$ is isomorphic to $\mathcal{A}(\circ_{CR}, f)$ for any $f \in (0, \frac{1}{2})$; e.g. for $f = \frac{1}{3}$ we use the following mapping (for other $f$s just use a different suitable basis of the logarithm):

$$h\colon [0,1] \to \overline{\mathbb{R}} \qquad \text{defined as} \qquad h(x) = \begin{cases} 1 + \log_2(\frac{x}{1-x}) & \text{if } x \in (0,1) \\ -\infty & \text{if } x = 0 \\ +\infty & \text{if } x = 1 \end{cases}$$

Interestingly enough, up to our knowledge, the logic of $\boldsymbol{C}$ has never been explored (however, the related logic CRL of $\mathcal{A}(\circ_{CR}, \frac{1}{2})$, which conflates the interpretation of $\mathtt{t}$ and $\mathtt{f}$, a clearly an undesired law, has been studied in [5]).

The goal of this contribution is to motivate the logic of $\boldsymbol{C}$ and present its basic mathematical properties in the customary manner of fuzzy logics [4].

# References


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