# Quantificational issues in Prawitzian validity

Antonio Piccolomini d'Aragona

University of Siena
Aix-Marseille Univ, CNRS, Centre Granger, Aix-en-Provence, France
`antonio.piccolomini@unisi.it, antonio.piccolomini-d-aragona@univ-amu.fr`

While standard model-theoretic semantics explains meaning and validity in terms of truth-preservation under interpretations over - mainly set-theoretic - structures, constructive semantics focuses on provability-preservation over relevant deductive - usually atomic - systems.

Prawitz's semantics is a constructive setup where proofs are understood as valid arguments - see mainly [6] - or epistemic grounds - see mainly [8]. It stems from Prawitz's normalisation theorems in Gentzen's natural deduction [7], and in particular from what Schroeder-Heister [11] called the "fundamental corollary" of Prawitz's results: closed derivations in suitable systems - e.g. intuitionistic logic - reduce to *canonical* form, that is, to closed derivations ending by an introduction. This may confirm Gentzen's [3] claim that introductions define the meaning of the logical constants, whereas eliminations can be shown to be unique functions of the introductions via reductions such as those used by Prawitz for proving normalisation, e.g.

$$\dfrac{\dfrac{\begin{matrix}[\alpha]\\ \mathscr{D}_1\\ \beta\end{matrix}}{\alpha\to\beta}\,(\to_I)\qquad \begin{matrix}\mathscr{D}_2\\ \alpha\end{matrix}}{\beta}\,(\to_E) \qquad \overset{\phi^{\to}}{\Longrightarrow}\qquad \begin{matrix}\mathscr{D}_2\\ [\alpha]\\ \mathscr{D}_1\\ \beta\end{matrix}$$

or, in Curry-Howard equivalent $\lambda$-form,

$$\texttt{App}(\lambda x.T(x), U) = T(U) \tag{1}$$

for $x, U : \alpha$ and $T : \beta$. In Prawitz's semantics, the "fundamental corollary" becomes Dummett's [2] *fundamental assumption*: if $\alpha$ is provable, then $\alpha$ is canonically provable. This implies moving from derivations to proofs, i.e. abstracting from specific systems, and allowing argument structures or linear forms to involve (types for) arbitrary inferences. So, reductions/equations act upon maximal formulas or redexes *latu sensu*, e.g. for disjunctive syllogism

$$\dfrac{\begin{matrix}\mathscr{D}_1\\ \alpha\lor\beta\end{matrix}\qquad \begin{matrix}\mathscr{D}_2\\ \neg\alpha\end{matrix}}{\beta}\,(DS)\qquad \overset{\phi^{DS}}{\Longrightarrow}\qquad \dfrac{\begin{matrix}\mathscr{D}_1\\ \alpha\lor\beta\end{matrix}\qquad \dfrac{\dfrac{[\alpha]\qquad \begin{matrix}\mathscr{D}_2\\ \neg\alpha\end{matrix}}{\bot}\,(\to_E)}{\beta}\,(\bot)\qquad [\beta]}{\beta}\,(\lor_E)$$

or, in a Curry-Howard alternative linear form,

$$DS(\texttt{inj}_2^{\alpha}(T), U) = T \tag{2}$$

for $T : \beta$ and $U : \neg\alpha$. The interpretation of the non-logical signs is attained by introducing atomic systems $\Sigma$, i.e. sets of (production) rules

$$\dfrac{\alpha_1, ..., \alpha_n}{\beta}$$

where $\alpha_i \neq \bot$ and $\beta$ are atomic ($i \leq n$) - plus other restrictions. Given an argument structure or linear form $\tau$ with (types for) arbitrary inferences, and a set $\mathscr{J}$ of reductions/equations for maximal formulas/redexes *latu sensu*, one then says that $\langle \tau, \mathscr{J} \rangle$ is, respectively, a valid argument or ground over $\Sigma$ iff $\tau$ reduces through $\mathscr{J}$ to a canonical form whose immediate substructures or sub-terms are, respectively, valid arguments or grounds over $\Sigma$ - possibly under closure of unbound variables and assumptions. An inference (rule) $R$ can be said to be valid on $\Sigma$ iff there is (a reduction/equation defining) a function $\phi$ such that, for every suitable valid argument or ground $\langle \tau, \mathscr{J} \rangle$ over $\Sigma$, $\langle \tau^*, \mathscr{J} \cup \{\phi\} \rangle$ is valid over $\Sigma$, where $\tau^*$ is the result of appending $R$ to $\tau$.

Prawitz's semantics has been often understood as a natural formal framework for Dummett's verificationist theory of meaning [2]. Dummett's anti-realist arguments seem to imply that intuitionistic logic - in short, IL - is the "correct" logic. Therefore, intuitionistic logic is expected to be complete with respect to Prawitz's semantics. This is what *Prawitz's conjecture* claims. But logical validity in Prawitz's framework can be defined in two ways, depending on the order in which we quantify over reductions/equations and atomic systems.

The first possibility is *system-rooted* validity - in short, PS-validity. $R$ is PS-valid iff, for every atomic system $\Sigma$, there is (a reduction/equation defining) a function $\phi$ such that $R$ is valid over $\Sigma$. The second possibility is *schematic* validity - in short, P-validity. $R$ is P-valid iff there is (a reduction/equation defining) a function $\phi$ such that, for every atomic system $\Sigma$, $R$ is valid over $\Sigma$. This can be extended to a more traditional relation between (sets of) formulas, i.e. $\Gamma \models_{\texttt{PS/P}} \alpha$ iff there is a PS/P-valid inference from $\Gamma$ to $\alpha$. Given the easily provable correcteness result with respect to both notions, i.e. $\Gamma \vdash_{\texttt{IL}} \alpha \Rightarrow \Gamma \models_{\texttt{PS/P}} \alpha$, Prawitz's conjecture then claims that the inverse also holds, i.e. $\Gamma \models_{\texttt{PS/P}} \alpha \Rightarrow \Gamma \vdash_{\texttt{IL}} \alpha$. Of course, whether the conjecture is true or not may vary depending on whether we choose PS- or P-validity.

In fact, Piecha and Schroeder-Heister [5] have proved that, if by logical validity we mean PS, Prawitz's conjecture fails - Harrop's rule being a counterexample. The importance of Piecha and Schroeder-Heister's proof does not only rely upon *what* it shows, but also on the fact that - contrarily to previous approaches, see e.g. [1] - it abstracts from specific restrictions on atomic systems - e.g. from whether we allow bindings at the atomic level, or higher-level atomic rules, see e.g. [10, 9]. Piecha and Schroeder-Heister introduce a number of principles which are shown to be sufficent for framing various notions of consequence, and for classifying various necessary or sufficient conditions of constructive (in)completeness.

However, some of these principles seem to crucially fail over P-validity, e.g. a sort of semantic *admissibility principle* $\Gamma \models_{\texttt{P}} \alpha \Leftrightarrow (\models_{\texttt{P}} \Gamma \Rightarrow \models_{\texttt{P}} \alpha)$, and a sort of semantic *disjunction property* $\Gamma \models_{\texttt{P}} \alpha \vee \beta \Leftrightarrow (\Gamma \models_{\texttt{P}} \alpha$ or $\Gamma \models_{\texttt{P}} \beta)$ for $\vee, \exists$ not occurring in $\Gamma$. So, Piecha and Schroeder-Heister's proof may not apply to P-validity, and Prawitz's conjecture would remain open.

In my talk, I first of all aim at highlighting the (often overlooked) distinction between PS- and P-validity relative to Piecha and Schroeder-Heister's proof. Based on this primary goal, I then address two further issues, leading to two derived notions of prawitzian validity.

First, what I call *choice validity* - in short, C-validity. This simply amounts to the idea that we can "extract" P-validity from PS-validity by allowing a choice-function $\mathscr{F}$ in the class of our reductions/equations. Suppose $\Gamma \models_{\texttt{PS}} \alpha$, then there is a PS-valid inference $R$ from $\Gamma$ to $\alpha$, then for every atomic system $\Sigma$ we can find (a reduction/equation defining) a function $\phi_\Sigma$ such that $R$ is valid over $\Sigma$. Thus, given any atomic system $\Sigma$, and any valid arguments or grounds $\mathscr{D}_\Sigma$ over $\Sigma$ for the elements of $\Gamma$, we may state

$$\begin{array}{c} \mathscr{D}_\Sigma \\ \dfrac{\Gamma}{\alpha}\, R \end{array} \quad \stackrel{\mathscr{F}}{\Longrightarrow} \quad \phi_\Sigma(\mathscr{D}^*_\Sigma)$$

where $\mathscr{D}^*_\Sigma$ is the result of appending $R$ to $\mathscr{D}_\Sigma$. In other words, $\mathscr{F}$ "picks" the right function for $R$ relative to the system where the input lives. Thus $\Gamma \models_{\mathtt{PS}} \alpha \Rightarrow \Gamma \models_{\mathtt{C}} \alpha$, and the claim would now be that, since we always use "one and the same" reduction/equation defining $\mathscr{F}$ to validate $R$ over all atomic systems, this is "sufficiently schematic" for having $\Gamma \models_{\mathtt{C}} \alpha \Rightarrow \Gamma \models_{\mathtt{P}} \alpha$. This would mean that Prawitz's conjecture is refuted also over P-validity. But a crucial difference occurs between $\mathscr{F}$ and $\phi^{\rightarrow}$ or $\phi^{DS}$. The latter are specified without referring to atomic systems and functions over these systems, whilst in the former these parameters must explicitly occur. So $\mathscr{F}$ is a function, not only of valid arguments or grounds, *but also of atomic systems.* This is much more evident if we move to a Curry-Howard linear form, say

$$\mathscr{F}(T, \Sigma) = \mathtt{h}(\Sigma)(T) \tag{3}$$

for $T : \Gamma$ and $\mathtt{h} : \mathtt{S} \hookrightarrow \mathtt{E}$, where S is the class of atomic systems and E is the class of functions for $R$ over elements of S - so h is our choice function. Contrarily to equations (1) and (2), equation (3) checks where its proof-input comes from, thus involving an additional parameter $\Sigma$ and a kind of "meta-function" from systems to functions for $R$. Its proof-output depends not only on the proof-input, but also on the domain of this proof-input.

Choice may be *in principle* acceptable, but two problems seem to affect our specific case here. First, the overall class of atomic systems may not be "sufficiently constructive" to allow for an acceptable usage of Choice. Secondly, when one thinks of schematic reductions/equations, one seemingly thinks of functions which only operate on proof-inputs, and which only generate proof-outputs, without additionally operating on the whole systems which these inputs and outputs belong to. If one accepts these objections, one should also reject the implication from C-validity to P-validity: additional restrictions should be put on schematic reductions/equations for them to respect what we mean by P-validity.

A truly fine-grained notion of P-validity requires specifying in greater detail what "schematic" means, i.e. defining more precisely the class of schematically acceptable reductions/equations. We know that a reduction/equation for $R$ must be such that the function $\phi$ associated to $R$ is linear over substitutions - i.e. $\phi(x[\star/\bullet]) = (\phi(x))[\star/\bullet]$ - and yields an output with the same type as, and no more variables and assumptions than the input. Given our previous discussion about C-validity, we know we must also have some limitations on the kind of inputs, whose range should be somehow bound to proof-objects, and exclude (functions on) structures where proof-objects live. But that said, it is anything but clear whether and when our restrictions-list can be said to be complete, nor is it clear how to formulate the restrictions in a rigorous way.

My proposal is that "schematic" is understood as "provably valid through logic and general proof-principles only", i.e.: that the reduction/equation for $R$ is schematic means that we can *prove*, with *no other means* than logic and general proof-principles, that such reduction/equation defines a function $\phi$ which validates $R$. No linguistic component referring to atomic systems can occur in such a proof, and in particular no such linguistic component can occur in the reduction/equation defining $\phi$. The proof goes through for every atomic system, without speaking of any system; hence, the same holds for the reduction/equation at issue.

For this proposal to make sense, we need systems where facts about inferences and reductions/equations can be proved. To this end, I build upon a class of systems for epistemic grounding that I introduced elsewhere [4], and define what I call *minimal grounding systems* - in short, MGS. An MGS relies upon a multi-sorted language where one can quantify over proofs, and contains:

- Gentzen's introductions and eliminations;
- generalised eliminations that fix an equational constraint for proofs of $\alpha$ to reduce to canonical proofs of $\alpha$ - namely, Dummett's fundamental assumption;

- syntactically schematic equations for eliminating redexes *latu sensu*, like (1) and (2) above.

This leads to the following definition of provable validity - in short, PR-validity. Let $R$ be an inference (rule) from assumptions $\alpha_1, ..., \alpha_n$ to $\beta$. Then, $R$ is PR-valid iff there is an equation $\varepsilon$ defining a function $\phi$ for $R$ such that, given an MGS containing $\varepsilon$ we have

$$\vdash_{\texttt{MGS}} \forall x_1 ... x_n (x_1 : \alpha_1 \wedge ... \wedge x_n : \alpha_n \rightarrow \phi(x_1, ..., x_n) : \beta) \quad (4)$$

- where some apparent variables $x_i$ may need to be replaced by functional variables $\texttt{h}_i$ taking as arguments variables $y$ (for assumptions $\gamma$) bound by $R$ on index $i$, in which case $x_i : \alpha_i$ in the antecedent is replaced by $\forall y_1 y_2 (y_2 : \gamma \rightarrow \texttt{h}_i(y_1, y_2) : \alpha_i)$ $(i \leq n)$. This definition raises some issues, with which I conclude my talk:

- PR-validity requires limiting to linear forms, which may be unproblematic insofar as valid arguments and reductions can be Curry-Howard translated to grounds and equations. But, given we are reasoning in extended frameworks, how can we grant that we have a full equivalence between the two approaches?
- Which logic should we choose for MGS? Does this influence PR-completeness?
- If we look at the MGS-s as a class, we can seemingly order rules through an order-relation $\Re$, and set a rank for sequences of rules in $\Re$, say $R$ is of degree 0 iff (4) is obtained in an MGS with no other equations than that for $R$, and it is of degree $i+1$ iff the MGS where (4) is proved is one containing equations for rules whose highest degree is $i$. This may in turn permit to reformulate Prawitz's conjecture, say IL is PR-complete iff, for each PR-valid $R$, there is an $\Re$-path ending with $R$ and whose mininimal elements are rules of IL.

# References


[1] W. de Campos Sanz, T. Piecha, and P. Schroeder-Heister. Constructive semantics, admissibility of rules and the validity of Peirce's law. *Logic journal of the IGPL*, 2013.

[2] M. Dummett. *The logical basis of metaphysics.* 1991.

[3] G. Gentzen. Untersuchungen über das Logische Schließen. *Matematische Zeitschrift*, 1934-1935.

[4] A. Piccolomini d'Aragona. Calculi of epistemic grounding based on prawitz's theory of grounds. *Studia Logica*, 2021.

[5] T. Piecha and P. Schroeder-Heister. Incompleteness of intuitionistic logic with respect to proof-theoretic semantics. *Studia Logica*, 2018.

[6] D. Prawitz. Towards a foundation of a general proof-theory. In P. Suppes, L. Henkin, A. Joja, and G. C. Mosil, editors, *Proceedings of the Fourth International Congress for Logic, Methodology and Philosophy of Science, Bucarest 1971*, 1973.

[7] D. Prawitz. *Natural deduction. A proof-theoretical study.* 2006.

[8] D. Prawitz. Explaining deductive inference. In H. Wansing, editor, *Dag Prawitz on proofs and meaning.* 2015.

[9] P. Schroeder-Heister. Generalized rules for quantifiers and the completeness of the intuitionistic operators $\wedge$, $\vee$, $\rightarrow$, $\forall$ and $\exists$. In M. M. Richter, E. Börger, W. Oberschelp, B. Schinzel, and W. Thomas, editors, *Computation and Proof Theory. Proceedings of the Logic Colloquium held in Aachen, July 18-23, 1983, Part II*, 1984.

[10] P. Schroeder-Heister. A natural extension of natural deduction. *Journal of philosophical logic*, 1984.

[11] P. Schroeder-Heister. Validity concepts in proof-theoretic semantics. *Synthese*, 2006.