# Polyhedral Completeness of Intermediate and Modal Logics

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In this talk we investigate a recent semantics for modal and intermediate logics using polyhedra. The starting point is that the collection of open subpolyhedra of a compact polyhedron (of any dimension) forms a Heyting algebra [1, 5, 7]. Precursors of this work are [2], [4] and [3].

Let $P$ be an $n$-dimensional compact polyhedron. By an *open subpolyhedron* of $P$ we mean a subset of $P$ whose complementary subset in $P$ is a compact polyhedron. Under inclusion order, the poset $\mathsf{Sub}(P)$ of all open subpolyhedra of $P$ is a Heyting algebra [5]. For a propositional formula $\varphi$, we say that $P \models \varphi$ if $\mathsf{Sub}(P) \models \varphi$ (i.e., $\varphi$ is valid in the Heyting algebra $\mathsf{Sub}(P)$). For a class $\mathcal{P}$ of polyhedra we write $\mathcal{P} \models \varphi$ if $P \models \varphi$ for each $P \in \mathcal{P}$.

In this abstract, we think of posets as both Kripke frames and topological spaces given by the Alexandrov topology of upwards-closed subsets. An important connector between polyhedra and posets is the notion of a polyhedral map. Let $P$ be a polyhedron and $F$ be a poset. A function $f\colon P \to F$ is a *polyhedral map* if the preimage of any open set in $F$ is an open subpolyhedron of $P$.

**Lemma 1.** *If $f\colon P \to F$ is polyhedral and open, then the logic of $P$ is contained in the logic of $F$.*

The purpose of this talk is to give a characterisation of the logic of *convex* polyhedra. We focus on the intermediate logic side in this abstract; analogous results hold for modal logic. Our logic $\mathsf{PL}$ is axiomatised by Jankov-Fine formulas. To every finite rooted poset $Q$, we associate a formula $\chi(Q)$, the *Jankov-Fine* formula of $Q$ (also called its *Jankov-De Jongh formula*). This has the property that $F \models \chi(Q)$ if and only if there is no surjective p-morphism $f$ from an open subset $U \subseteq F$ onto $Q$ [6, §9]. Our logic $\mathsf{PL}$ is then axiomatised by adding to intuitionistic propositional calculus the Jankov-Fine formulas of two simple posets:

$$\mathsf{PL} = \mathsf{IPC} + \chi(\text{}) + \chi(\text{})$$

**Theorem 2.** $\mathsf{PL}$ *is the logic of all convex polyhedra.*

Moreover, we obtain the more fine-grained characterisation of the logic of convex polyhedra of dimension at most $n$ by extending the logic of bounded depth $n$ [6, §9]. Let:

$$\mathsf{PL}_n = \mathsf{BD}_n + \chi(\text{}) + \chi(\text{})$$

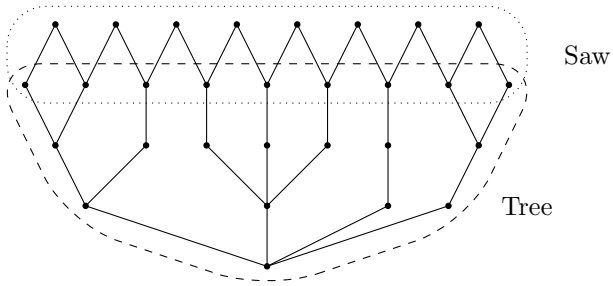


Figure 1: An example sawed tree

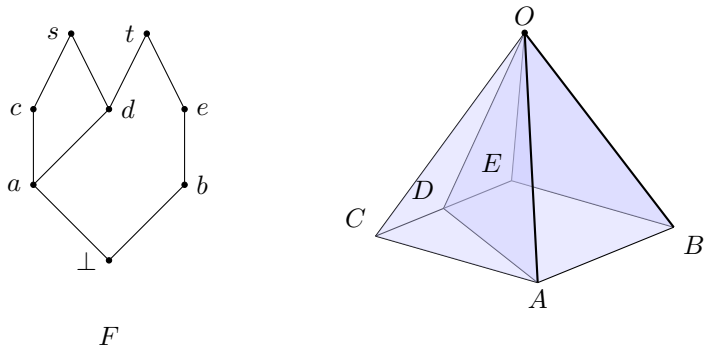


Figure 2: An example of convex geometric realisation

**Theorem 3.** $\mathsf{PL}_n$ *is the logic of all convex polyhedra of dimension at most* $n$.

We prove Theorem 3 first. This splits into the soundness and completeness direction. For the soundness direction, we show using a geometric argument that the logic that every convex $n$-dimensional polyhedron is the same, and that $\mathsf{PL}_n$ is valid on the simplest example of these: the $n$-simplex.

The completeness direction splits into two steps. In the first step, we show that any poset $F$ validating $\mathsf{PL}_n$ is the p-morphic image of a frame which has a special form, called a *sawed tree*. This consists of a planar tree of uniform height with a 'saw structure' on top. See Figure 1 for an example. In the second step, we show how to realise any sawed tree of height $n$ as a convex polyhedra. This convex polyhedron comes equipped with an open polyhedral map onto the original sawed tree. By Lemma 1, this means that the logic of the convex polyhedron is contained in the logic of $F$. See Figure 2 for an example of this process of geometric realisation.

Finally, with Theorem 3 established, we make use of a result of Zakharyaschev, which entails that $\mathsf{PL}$ is the logic of its finite frames (i.e. it has the finite model property) [8, Corollary 0.11]. This then completes the proof of Theorem 2.

# References

[1] S. Adam-Day, N. Bezhanishvili, D. Gabelaia, and V. Marra. Polyhedral completeness of intermediate logics: the nerve criterion, 2021.

[2] M. Aiello, J. van Benthem, and G. Bezhanishvili. Reasoning about space: the modal way. *J. Logic Comput.*, 13(6):889–920, 2003.

[3] J. van Benthem and G. Bezhanishvili. Modal logics of space. In *Handbook of Spatial Logics*, pages 217–298. Springer, Dordrecht, 2007.

[4] J. van Benthem, G. Bezhanishvili, and M. Gehrke. Euclidean hierarchy in modal logic. *Studia Logica*, 75(3):327–344, 2003.

[5] N. Bezhanishvili, V. Marra, D. Mcneill, and A. Pedrini. Tarski's theorem on intuitionistic logic, for polyhedra. *Annals of Pure and Applied Logic*, 169(5):373–391, 2018.

[6] A. Chagrov and M. Zakharyaschev. *Modal logic*, volume 35 of *Oxford Logic Guides*. The Clarendon Press, Oxford University Press, New York, 1997. Oxford Science Publications.

[7] D. Gabelaia, K. Gogoladze, M. Jibladze, E. Kuznetsov, and M. Marx. Modal logic of planar polygons. Preprint submitted to Elsevier, 2018.

[8] M. Zakharyaschev. A Sufficient Condition for the Finite Model Property of Modal Logics above K4. *Logic Journal of the IGPL*, 1(1):13–21, 07 1993.