Logic Beyond Formulas: Designing Proof Systems on Graphs

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Keeping track of relations between objects or events is essential in modeling processes and in verifying their security and privacy properties. For this purpose, relations are encoded by means of formulas in order to use proof theoretical results to design verification tools.

The "happens before" relation [12], providing a partial order between events to express when an event precede another, is crucial when studying distributed systems. Its restriction to series-parallel orders have received a special attention [16, 9, 5], giving rise to a family of non-commutative logics, including pomset logic [14] and BV [10].

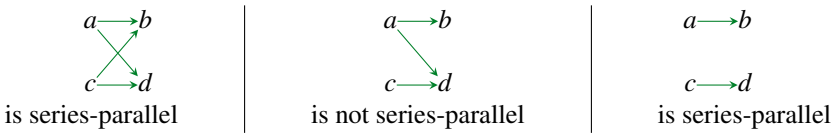
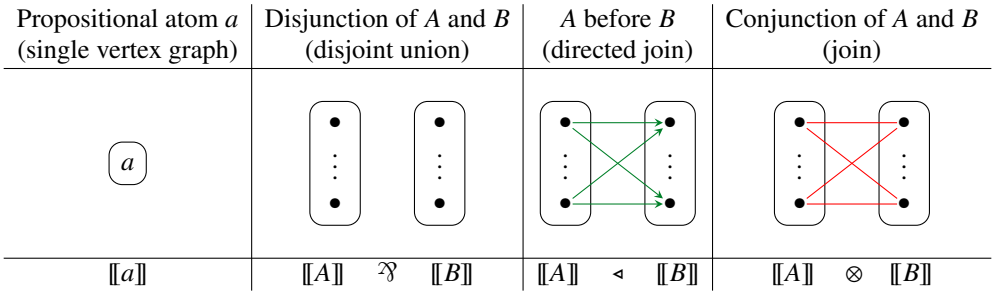


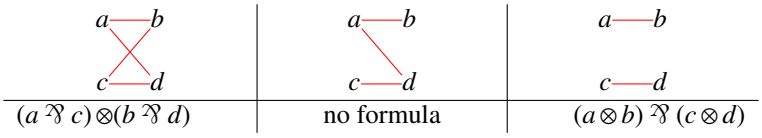
Figure 1: Three partial orders represented by their Hessse diagrams.

However, relations admitting no series-parallel decomposition [7] cannot be directly treated by the same proof theoretical methods since they require the use of encodings, which create a gap between meaning (semantics) and formal representation (syntax). In fact, the natural correspondence between graphs and formulas provided by the operations below [14, 10] fails as soon as simple topological conditions are not met in the graph representing a relation.



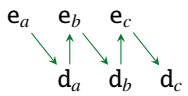
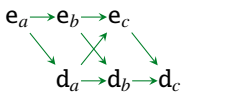
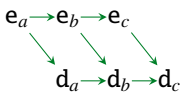
(1)

By means of example, consider four processes a, b, c and d where communication between some processes is forbidden because of certain conflicts of interest [6]. Thus, the following pairs cannot communicate: a and b, a and d, and c and d, as shown in the graph below in the center where the edges represent the impossibility of communication between processes.



(2)

Another example is given by the causality patterns for n-queues, where n is the bound on the number of elements that can be enqueued. These patterns can be represented by the graphs below, where nodes labelled by e_x and d_x respectively represent the enqueuing and dequeuing of the element x (we only represent the first three elements a, b, and c inserted into the queue), and edges represent the "happens before" relation. Among these graphs, only Q_1 and Q_2 are series-parallel graphs and can be directly encoded as formulas. In fact, the graph Q_3, and more in general the causality patterns for n-queues with $n > 2$, cannot.

a 1-queue Q_1	a 2-queue Q_2	a 3-queue Q_3
		
$\mathsf{e}_a \triangleleft \mathsf{d}_a \triangleleft \mathsf{e}_b \triangleleft \mathsf{d}_b \triangleleft \mathsf{e}_c \triangleleft \mathsf{d}_c$	$\mathsf{e}_a \triangleleft (\mathsf{d}_a \parr \mathsf{e}_b) \triangleleft (\mathsf{d}_b \parr \mathsf{e}_c) \triangleleft \mathsf{d}_c$	no formula

(3)

This contribution, based on joint works with Straßburger, Horne and Mauw [3, 2, 1], is an introduction on the proof theory of proof systems operating on graphs instead of formulas. This line of work aims at defining proof theoretical tools able to directly handle non series-parallel relations as primitive objects of a logic.

In order to design such systems, we use results on graph modular decomposition [13] allowing us to associate abstract syntax trees to graphs, and therefore to generalize the notions of connective and subformula which are fundamental to express desirable proof theoretical notions. After defining a (linear) implication $\multimap$, we define proof systems meeting certain basic desiderata such as the derivability of the general identity ($G \multimap G$ is provable for any graph G), the transitivity of implication (if $G \multimap H$ and $H \multimap K$ are provable, then $G \multimap K$ also is), and analyticity (if G is provable, then G admits a proof containing only its "subformulas" of G) . To this end, we use the open deduction [11] proof formalism (see Figure 2 for an example) based on deep inference [4] since, as observed for the non-commutative logic BV [10, 15], it is not possible to define an analytic sequent calculus for these logics.

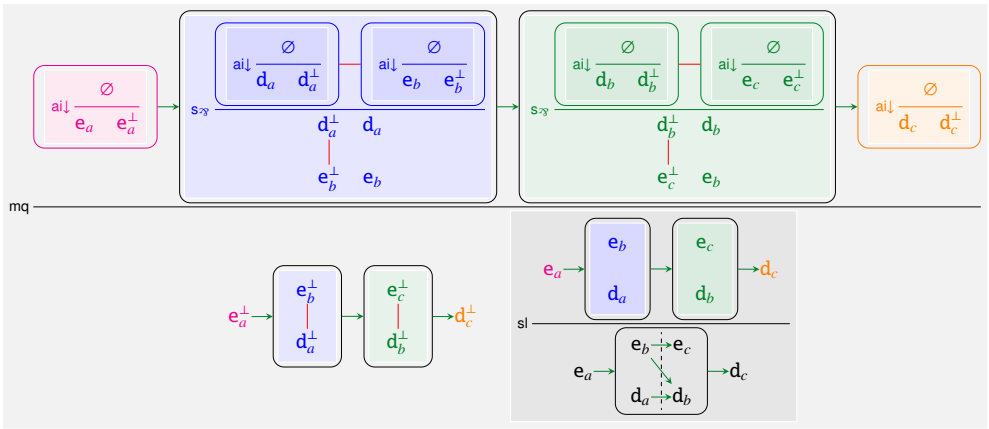


Figure 2: A proof of the graph $\mathsf{Q}_3 \multimap \mathsf{Q}_2$ in the system $\mathsf{GV}^{\mathsf{sl}}$ serving as proof that 3-queues can simulate behaviours of 2-queues. The rule sl slices a directed graph into a "before" and an "after" part by introducing additional directed edges. The rule mq merges the modules of two copies of the same directed graph.

We present the system GS, handling undirected graphs as the ones in Equation 2, and the systems GV and $\mathsf{GV}^{\mathsf{sl}}$ handling graphs with both directed and undirected edges. The system GS defines a conser-

vative extension of the *multiplicative linear logic with mix* [8], while the systems GV and GV$^{\mathsf{sl}}$ defines conservative extensions of both the graphical logic defined by GS and the non-commutative logic BV. We present the technique developed to prove these results, including the challenges we encountered in proving the analogous of cut-elimination for deep inference systems in the graphical setting. We conclude by recalling related results in proof theory and concurrency theory, their possible applications to verification thanks to their more expressive power, and giving an overview on the the ongoing researches on the topic.

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