Gödel temporal logic

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Abstract

We present investigations of a non-classical version of linear temporal logic (with next, eventually, and henceforth modalities) whose propositional fragment is Gödel–Dummett logic (which is well known both as a superintuitionistic logic and a t-norm fuzzy logic). We define the logic using two natural semantics—a real-valued semantics and a semantics where truth values are captured by a linear Kripke frame—and can show that these indeed define one and the same logic. Although this Gödel temporal logic does not have any form of the finite model property for these two semantics, we are able to prove decidability of the validity problem. The proof makes use of quasimodels, which are a variation on Kripke models where time can be nondeterministic. We can show that every falsifiable formula is falsifiable on a finite quasimodel, which yields decidability. We then strengthen this result to PSPACE-complete. Further, we provide a deductive calculus for Gödel temporal logic with a finite number of axioms and deduction rules, and can show this calculus to be sound and complete for the above-mentioned semantics.

1 Introduction

The importance of temporal logics and, independently, of fuzzy logics in computer science is well established. The potential usefulness of their combination is clear: for instance, it would provide a natural framework for the specification of programs dealing with vague data. Subclassical temporal logics have mostly been studied in the context of here-and-there logic, which allows for three truth values and is the basis for temporal answer set programming [1, 2, 3].

One may, however, be concerned that infinite-valued temporal logics could lead to an explosion in computational complexity, as has been known to happen when combining fuzzy logic with transitive modal logics: these combinations are often undecidable [11], or decidable with only an exponential upper bound being known [4]. As we will see, this need not be the case: the combination of Gödel–Dummett logic with linear temporal logic, which we call Gödel temporal logic (GTL), remains PSPACE-complete, the minimal possible complexity given that classical LTL embeds into it. This is true even when the logic is enriched with the dual implication [10], which has been argued in [5] to be useful for reasoning with incomplete or inconsistent information.

The decidability of GTL is already surprising, as it does not enjoy the finite model property. In fact, GTL possesses two natural semantics, corresponding to whether it is viewed as a fuzzy logic or a superintuitionistic logic. As a fuzzy logic, propositions take values in $[0, 1]$, and truth values of compound propositions are defined using standard operations on the real line. As a superintuitionistic logic, models consist of Kripke structures equipped with a partial order to

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interpret implication intuitionistically and a function to interpret the LTL tenses. Remarkably, the two semantics give rise to the same set of valid formulas.

To overcome the failure of the finite model property, we introduce quasimodels, which do enjoy their own version of the finite model property. Quasimodels are not 'true' models in that the functionality of the 'next' relation is lost, but they give rise to standard Kripke models by unwinding. Similar structures were used to prove upper complexity bounds for dynamic topological logic [6, 7] and intuitionistic temporal logic [8], but they are particularly effective in the setting of Gödel temporal logic, as they yield an optimal PSPACE upper bound.

Finally, we provide a deductive calculus for Gödel temporal logic with a finite number of axioms and deduction rules, and can show this calculus to be sound and complete for the above-mentioned semantics.

2 Syntax and semantics

Fix a countably infinite set $\mathbb{P}$ of propositional variables. Then the **Gödel temporal language** $\mathcal{L}$ is defined by the grammar (in Backus–Naur form):

$$\varphi, \psi := p \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \varphi \Rightarrow \psi \mid \varphi \Leftarrow \psi \mid \bigcirc\varphi \mid \Diamond\varphi \mid \Box\varphi,$$

where $p \in \mathbb{P}$. Here, $\bigcirc$ is read as 'next', $\Diamond$ as 'eventually', and $\Box$ as 'henceforth'. The connective $\Leftarrow$ is coimplication and represents the operator dual to implication [12]. We also use $\bot$ as a shorthand for $p \Leftarrow p$ and $\neg\varphi$ as a shorthand for $\varphi \Rightarrow \bot$.

We now introduce the first of our semantics for the Gödel temporal language: real semantics, which views $\mathcal{L}$ as a fuzzy logic (enriched with temporal modalities). In the definition, $[0, 1]$ denotes the real unit interval.

Definition 1 (real semantics)**.** A **flow** is a pair $\mathcal{T} = (T, S)$, where T is a set and $S\colon T \to T$ is a function. A **real valuation** on $\mathcal{T}$ is a function $V\colon \mathcal{L} \times T \to [0, 1]$ such that, for all $t \in T$, the following equalities hold.

$$\begin{array}{rcl@{\qquad}rcl}
V(\varphi \wedge \psi, t) & = & \min\{V(\varphi, t)V(\psi, t)\} & V(\varphi \vee \psi, t) & = & \max\{V(\varphi, t), V(\psi, t)\} \\
V(\varphi \Rightarrow \psi, t) & = & \begin{cases} 1 \text{ if } V(\varphi, t) \leq V(\psi, t) \\ V(\psi, t) \text{ otherwise} \end{cases} & V(\varphi \Leftarrow \psi, t) & = & \begin{cases} 0 \text{ if } V(\varphi, t) \leq V(\psi, t) \\ V(\varphi, t) \text{ otherwise} \end{cases} \\
\multicolumn{6}{c}{V(\bigcirc\varphi, t) = V(\varphi, S(t))} \\
V(\Diamond\varphi, t) & = & \sup_{n<\omega} V(\varphi, S^n(t)) & V(\Box\varphi, t) & = & \inf_{n<\omega} V(\varphi, S^n(t))
\end{array}$$

A flow $\mathcal{T}$ equipped with a valuation V is a **real (Gödel temporal) model**.

The second semantics, Kripke semantics, views $\mathcal{L}$ as an intuitionistic logic (temporally enriched). Below, define $\vec{S}(w, t) = (w, S(t))$.

Definition 2 (Kripke semantics)**.** A **(Gödel temporal) Kripke frame** is a quadruple $\mathcal{F} = (W, T, \leq, S)$ where $(W, \leq)$ is a linearly ordered set and (T, S) is a flow. A **Kripke valuation** on $\mathcal{F}$ is a function $[\![\cdot]\!] : \mathcal{L} \to 2^{W \times T}$ such that, for each $p \in \mathbb{P}$, the set $[\![p]\!]$ is *downward closed* in its first coordinate, and the following equalities hold.

$$\begin{array}{c}
[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \cap [\![\psi]\!] \qquad [\![\varphi \vee \psi]\!] = [\![\varphi]\!] \cup [\![\psi]\!] \\
[\![\varphi \Rightarrow \psi]\!] = \{(w, t) \in W \times T \mid \forall v \leq w((v, t) \in [\![\varphi]\!] \text{ implies } (v, t) \in [\![\psi]\!])\} \\
[\![\varphi \Leftarrow \psi]\!] = \{(w, t) \in W \times T \mid \exists v \geq w((v, t) \in [\![\varphi]\!] \text{ and } (v, t) \notin [\![\psi]\!])\} \\
[\![\bigcirc\varphi]\!] = \vec{S}^{-1}\,[\![\varphi]\!] \qquad [\![\Diamond\varphi]\!] = \bigcup_{n<\omega} \vec{S}^{-n}\,[\![\varphi]\!] \qquad [\![\Box\varphi]\!] = \bigcap_{n<\omega} \vec{S}^{-n}\,[\![\varphi]\!]
\end{array}$$

A Kripke frame $\mathcal{F}$ equipped with a valuation $[\![\cdot]\!]$ is a **(Gödel temporal) Kripke model**.

Definition 3 (validity)**.** A formula φ is **valid** with respect to the real semantics if $V(\varphi, t) = 1$ at all times t in all real models, otherwise φ if **falsifiable**.

A formula φ is **valid** with respect to the Kripke semantics if $[\![\varphi]\!] = W \times T$ in all Kripke models, otherwise φ if **falsifiable**.

We define the logic $\mathsf{GTL}_{\mathbb{R}}$ to be the set of $\mathcal{L}$-formulas that are valid over the class of all flows and the logic $\mathsf{GTL}_{\mathrm{K}}$ to be the set of $\mathcal{L}$-formulas that are valid over the class of all Kripke frames.

3 Results

Using model-theoretic arguments, we can prove the following result.

Theorem 1. *Validity over real and Kripke semantics coincide, that is:* $\mathsf{GTL}_{\mathbb{R}} = \mathsf{GTL}_{\mathrm{K}}$.

We now turn to the question of decidability/complexity of this set of validities. As we mentioned, finite model properties fail; we now make this precise.

Definition 4. The **strong finite model property** is the statement that if $\varphi \in \mathcal{L}$ is falsifiable on a Kripke model, then it is falsifiable on a Kripke model $\mathcal{F} = (W, T, \leq, S, [\![\cdot]\!])$ where both W and T are finite.

The **order finite model property** is the statement that if $\varphi \in \mathcal{L}$ is falsifiable on a Kripke model, then it is falsifiable on a Kripke model $\mathcal{F} = (W, T, \leq, S, [\![\cdot]\!])$ where W is finite.

The **temporal finite model property** is the statement that if $\varphi \in \mathcal{L}$ is falsifiable on a Kripke model, then it is falsifiable on a Kripke model $\mathcal{F} = (W, T, \leq, S, [\![\cdot]\!])$ where T is finite.

Proposition 2. *None of the finite model properties listed in Definition 4 hold. In particular,* $\Diamond(p \Rightarrow \bigcirc p)$ *is falsifiable, yet it is valid over the class of finite Kripke models.*

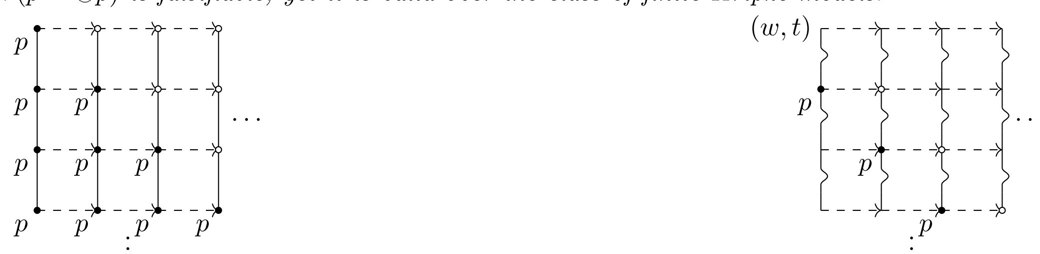

Figure 1: Left: A Kripke model falsifying $\Diamond(p \Rightarrow \bigcirc p)$; right: W and T are necessarily infinite.

However, by defining and utilising *quasimodels*, we can prove the following.

Theorem 3. *The decision problem of testing validity for* GTL *is decidable.*

Theorem 4. *The decision problem of testing validity for* GTL *is* PSPACE*-complete.*

Finally, we prove the soundness and completeness of the following deductive system.

1. **All (substitution instances of) intuitionistic tautologies**
2. **Axioms and rules of H-B logic (cf. [9]):**

$$\varphi \Rightarrow (\psi \vee (\varphi \Leftarrow \psi)) \qquad \frac{\varphi \Rightarrow \psi}{(\varphi \Leftarrow \theta) \Rightarrow (\psi \Leftarrow \theta)} \qquad \frac{\varphi \Rightarrow \psi \vee \gamma}{(\varphi \Leftarrow \psi) \Rightarrow \gamma}$$

3. **Linearity axioms:** $(\varphi \Rightarrow \psi) \vee (\psi \Rightarrow \varphi) \qquad \neg((\varphi \Leftarrow \psi) \wedge (\psi \Leftarrow \varphi))$
4. **Temporal axioms:**

(a) $\neg \bigcirc \bot$

(b) $\bigcirc (\varphi \vee \psi) \Rightarrow (\bigcirc\varphi \vee \bigcirc\psi)$

(c) $(\bigcirc\varphi \wedge \bigcirc\psi) \Rightarrow \bigcirc (\varphi \wedge \psi)$

(d) $\bigcirc (\varphi \Rightarrow \psi) \Leftrightarrow (\bigcirc\varphi \Rightarrow \bigcirc\psi)$

(e) $\Box (\varphi \Rightarrow \psi) \Rightarrow (\Box\varphi \Rightarrow \Box\psi)$

(f) $\Box (\varphi \Rightarrow \psi) \Rightarrow (\Diamond\varphi \Rightarrow \Diamond\psi)$

(g) $\Box\varphi \Rightarrow \varphi \wedge \bigcirc\Box\varphi$

(h) $\varphi \vee \bigcirc\Diamond\varphi \Rightarrow \Diamond\varphi$

(i) $\Box(\varphi \Rightarrow \bigcirc\varphi) \Rightarrow (\varphi \Rightarrow \Box\varphi)$

(j) $\Box(\bigcirc\varphi \Rightarrow \varphi) \Rightarrow (\Diamond\varphi \Rightarrow \varphi)$

5. **Back–up confluence axiom:** $\bigcirc (\varphi \Leftarrow \psi) \Rightarrow (\bigcirc\varphi \Leftarrow \bigcirc\psi)$

6. **Standard modal rules:**

(a) $\dfrac{\varphi,\ \varphi \Rightarrow \psi}{\psi}$ (b) $\dfrac{\varphi}{\bigcirc\varphi}$ (c) $\dfrac{\varphi}{\Box\varphi}$

Theorem 5. *The smallest set of $\mathcal{L}$-formulas closed under the above axioms and rules is the set* $\mathsf{GTL}_{\mathbb{R}}$ $(= \mathsf{GTL}_{\mathrm{K}})$ *of Gödel temporal logic validities.*

The proof works by building a canonical quasimodel falsifying a given unprovable formula.

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