Nelson conuclei and nuclei: the twist construction beyond involutivity

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Until recently, twist-products (also known, in the literature, as *twist-structures* or *twist-algebras*) have been almost exclusively employed to construct and represent algebras (of non-classical logics) that carried an *involutive* negation, i.e. one satisfying the double negation identity ($\sim\sim x = x$). Prominent examples include various classes of bilattices and residuated structures, such as *Nelson algebras* (models of Nelson's constructive logic with strong negation: see e.g. [19]) and *N4-lattices* (models of the paraconsistent version of Nelson's logic: see [6, 7]). While the twist-product indeed provides an easy way to introduce an involutive negation, this feature is not essential to the construction, either from a technical or a conceptual point of view (concerning this latter aspect, see in particular [4, 5]). This observation is developed in a series of recent papers which explore the applicability of various non-involutive twist-product constructions: see [9, 8, 10, 11, 12, 13, 14, 15, 16, 17, 18]. In particular, the papers [18, 11] show how to generalize the definitions and twist representations of Nelson algebras and N4-lattices to a non-involutive setting; the resulting classes of algebras have been dubbed *quasi-Nelson algebras* and *quasi-N4-lattices*, respectively.

In the present contribution we extend the non-involutive twist construction so as to encompass yet more general classes of algebras. Observing that both Nelson and quasi-Nelson algebras are (special subclasses of) bounded commutative integral distributive residuated lattices, it seems natural to look at which among these properties (e.g. commutativity, integrality, distributivity) may be relaxed without compromising the twist representation. We accomplish our task considering two different approaches to twist-products: the original approach due to Rasiowa and the more recent one based on residuated lattices.

The algebraic models of Nelson's logic were first introduced (by H. Rasiowa, after D. Nelson's original presentation of the logic) in a language which featured a (non-residuated) intuitionistic-like implication (known in the literature as *weak implication*, and usually denoted by $\to$), and were later shown to be term equivalent to a class of integral residuated lattices. In the setting of Nelson algebras no problem arises, for the neutral element of the semigroup operation – which, by integrality, coincides with the top element of the associated lattice order – is a term definable algebraic constant; in fact, Nelson algebras can be characterized as precisely those N4-lattices on which the defining term ($x \to x$) is constant. In the non-integral setting of N4-lattices, however, the neutral element is no longer definable (such an element may not exist at all), and must therefore be introduced through a primitive nullary operation – if one wishes, that is, to study the models of paraconsistent Nelson's logic within the theory of residuated lattices. The class of N4-lattices enriched with such an extra constant is investigated in [1] under the

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name of *eN4-lattices* (another paper [3] uses the more suggestive name of *Nelson paraconsistent lattices*)[1].

This problem necessarily carries over to more general algebras. The paper [2] introduced a twist construction which determines a class of residuated lattices (dubbed *Kalman lattices*) that are commutative and involutive, but not necessarily integral nor distributive. The Kalman lattices of [2] include as subvarieties eN4-lattices but not N4-lattices[2].

The preceding considerations entail that the two approaches to twist-products – the one based on the strong implication (by which one obtains residuated structures) and the one based on the weak implication (which generalizes directly Rasiowa's construction of Nelson algebras) – grow further apart as we consider more general structures. This, in turn, suggests that it may be appropriate to pursue both approaches separately. In the present contribution we shall do so, drawing inspiration directly from the twist constructions presented in [2, 3] and extending them to a non-involutive setting. In particular, as far as the residuated lattice approach is concerned, we shall generalize the *Kalman lattices* of [2] and the *Nelson conucleus algebras* of [3] by simultaneously dropping the requirements of (i) involutivity of the negation, (ii) commutativity of the monoid operation (thus we shall work with *two* residuated implications, the left and right residuals of the monoid operation), and (iii) integrality of the factor algebras employed in the twist construction. With regards to the other approach, we shall generalize the *Rasiowa-type algebras* of [3] (i) by allowing the negation to be non-involutive and (ii) by not postulating the existence of a neutral element for the semigroup operation; as in the preceding case, here too we shall be dealing with two ("weak") implications.

References

[1] M. Busaniche and R. Cignoli. Residuated lattices as an algebraic semantics for paraconsistent Nelson's logic. *Journal of Logic and Computation*, 19(6):1019–1029, 2009.

[2] M. Busaniche and R. Cignoli. The subvariety of commutative residuated lattices represented by twist-products. *Algebra Universalis*, 71(1):5–22, 2014.

[3] M. Busaniche, N. Galatos and M.A. Marcos. Twist structures and Nelson conuclei. *Studia Logica* , 110: 949–987, 2022.

[4] T. Jakl, A. Jung and A. Pultr. Bitopology and four-valued logic. *Electronic Notes in Theoretical Computer Science*, 325:201–219, 2016.

[5] A. Jung, P. Maia and U. Rivieccio. Non-involutive twist-structures. Submitted to the *Logic Journal of the IGPL*, Special Issue on *Recovery Operators and Logics of Formal Consistency and Inconsistencies*, 28 (5), 2020, pp. 973–999.

[6] S. P. Odintsov. Algebraic semantics for paraconsistent Nelson's logic. *Journal of Logic and Computation*, 13(4):453–468, 2003.

[7] S. P. Odintsov. On the representation of N4-lattices. *Studia Logica*, 76(3):385–405, 2004.

[8] U. Rivieccio. Fragments of Quasi-Nelson: The Algebraizable Core. *Logic Journal of the IGPL*, DOI: 10.1093/jigpal/jzab023.

[9] U. Rivieccio. Fragments of Quasi-Nelson: Residuation. Submitted.

[1] As the twist construction shows, the impact of adding the extra constant is substantial, and in consequence both the twist representation and the term equivalence result are much more straightforward for eN4-lattices than for N4-lattices (see [20]).

[2] What is even worse, is that not even Nelson algebras can be viewed as a subvariety of eN4-lattices: this is because the definition of eN4-lattices implies (essentially for technical reasons) that the interpretation of e must be not only the neutral element of the monoid operation, but also a fixpoint of the negation, a requirement that no (non-trivial) Nelson algebra can satisfy.

[10] U. Rivieccio. Fragments of Quasi-Nelson: Two Negations. *Journal of Applied Logic*, 7: 499–559, 2020.

[11] U. Rivieccio. Quasi-N4-lattices. *Soft Computing*, 2022, DOI: 10.1007/s00500-021-06719-9.

[12] U. Rivieccio. Representation of De Morgan and (semi-)Kleene lattices. *Soft Computing*, 24 (12):8685–8716, 2020.

[13] U. Rivieccio and T. Flaminio. Prelinearity in (quasi-)Nelson logic. *Fuzzy Sets and Systems*, to appear.

[14] U. Rivieccio, T. Flaminio, and T. Nascimento. On the representation of (weak) nilpotent minimum algebras. In *2020 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE)*, pp. 1–8. Glasgow, United Kingdom, 2020. DOI: 10.1109/FUZZ48607.2020.9177641.

[15] U. Rivieccio and R. Jansana. Quasi-Nelson algebras and fragments. *Mathematical Structures in Computer Science*, 2021, DOI: 10.1017/S0960129521000049.

[16] U. Rivieccio, R. Jansana, and T. Nascimento. Two dualities for weakly pseudo-complemented quasi-Kleene algebras. In: Lesot M.J. et al. (eds), *Information Processing and Management of Uncertainty in Knowledge-Based Systems. IPMU 2020. Communications in Computer and Information Science*, vol. 1239, Springer, pp. 634-653, 2020.

[17] U. Rivieccio and M. Spinks. Quasi-Nelson algebras. *Electronic Notes in Theoretical Computer Science*, 344:169–188, 2019.

[18] U. Rivieccio and M. Spinks. Quasi-Nelson; or, non-involutive Nelson algebras. In D. Fazio, A. Ledda, F. Paoli (eds.), *Algebraic Perspectives on Substructural Logics* (Trends in Logic, 55), pp. 133–168, Springer, 2020.

[19] M. Spinks, U. Rivieccio, and T. Nascimento. Compatibly involutive residuated lattices and the Nelson identity. *Soft Computing* 23:2297–2320, 2019.

[20] M. Spinks, R. Veroff. Paraconsistent constructive logic with strong negation as a contraction-free relevant logic. In: J. Czelakowski (ed.) Don Pigozzi on Abstract Algebraic Logic, Universal Algebra, and Computer Science, *Outstanding Contributions to Logic*, vol. 16, pp. 323–379. Springer International Publishing, Switzerland (2018).