Degrees of FMP in extensions of bi-intuitionistic logic

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Abstract

We investigate degrees of FMP in extensions of bi-intuitionistic logic. Motivated by the proof of the intuitionistic case, we define a bi-intuitionistic version of the logic KG and restrict attention to its extensions. There we find useful properties of simple algebras and give a description of extensions with the FMP. Consequently, we provide a full characterisation of degrees of FMP, stating that the only existing degrees are 1 and $2^{\aleph_0}$, which is in stark contrast with the intuitionistic case.

1 Introduction

The notion of a degree of incompleteness, introduced in [5] (see also [4, Chapter 10.5]), is an important property in the theory of modal logic. It measures the cardinality of logics that have the same frames as a given logic. The study of degrees of incompleteness culminates in the result known as Blok's dichotomy [2], [3]. It states that every normal modal logic has degree of incompleteness 1 or $2^{\aleph_0}$.

This inspired the inception of another notion, called degree of the finite model property (degree of FMP for short) [1]. It counts the cardinality of logics that share the same finite frames as a given logic. Although this yields a similar definition, a striking difference appears when looking at the existing degrees of FMP of extensions of S4. As proven in [1], it turns out that in addition to 1 and $2^{\aleph_0}$, for every cardinal between 2 and $\aleph_0$ inclusive, there exists a logic of that degree. Moreover, the same characterisation holds for degrees of FMP of superintuitionistic logics. This gives motivation for the open problem of finding all existing degrees in extensions of bi-intuitionistic logic – a conservative extension of intuitionistic logic with an additional co-implication connective. Our work aims to shed light on this question by adapting the techniques used in the intuitionistic setting to the bi-intuitionistic setting.

2 The Kuznetsov-Gerĉiu logic

The characterisation of degrees of FMP of superintuitionistic logics follows from the explicit construction of logics with desired degrees. This construction takes advantage of the Kuznetsov-Gerĉiu logic KG [6] and its rich combinatorial properties. In order to define KG, one can make use of the following operation.

Definition 1. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be Heyting algebras. By the sum of $\boldsymbol{A}$ and $\boldsymbol{B}$ we mean the Heyting algebra obtained by placing $\boldsymbol{A}$ below $\boldsymbol{B}$ and identifying the top of $\boldsymbol{A}$ with the bottom of $\boldsymbol{B}$.

Recall that the free Heyting algebra on one generator is the Rieger-Nishimura lattice [8], [7]. Hence all 1-generated Heyting algebras are obtained as homomorphic images of the Rieger-Nishimura lattice.

Definition 2. The logic KG is defined as the logic of all finite sums of 1-generated Heyting algebras.

Crucially, for every cardinal κ between 1 and $\aleph_0$, there exists an extension of KG of degree κ. Therefore, we are interested in defining a suitable bi-intuitionistic analogue of KG and characterising possible degrees of its extensions.

3 The logic bi-KG

We look more closely at the definition of KG with the aim of adapting it to a bi-intuitionistic version. Notice that every 1-generated Heyting algebra can be uniquely equipped with a co-implication, thus becoming a bi-Heyting algebra. This allows us to introduce the following logic.

Definition 3. Let $\mathscr{G}$ be the class of all finite sums of 1-generated Heyting algebras, viewed as bi-Heyting algebras. The logic bi-KG is defined as the logic of $\mathscr{G}$.

It turns out that a very useful step towards understanding bi-KG is finding the universal class $\mathbb{U}(\mathscr{G}) = \mathbb{SP}_U(\mathscr{G})$. We accomplish this by describing precisely the local structure of algebras in $\mathscr{G}$ via universal formulas. Two algebras in $\mathbb{U}(\mathscr{G})$ are of particular interest – a Rieger-Nishimura variant that goes downward instead of upward and a Rieger-Nishimura variant that goes both upward and downward. Importantly, we have the following.

Theorem 4. Let $\mathscr{G}'$ be the collection $\mathscr{G}$ together with the two new Rieger-Nishimura variants. Then $\mathbb{U}(\mathscr{G})$ consists of all sums of algebras in $\mathscr{G}'$.

Since we know that these sums are always simple algebras[1], we deduce the following strong property.

Corollary 5. bi-KG is semi-simple and it is generated by $\mathbb{U}(\mathscr{G})$.

4 The FMP in extensions of bi-KG

The remainder of our work is dedicated to giving a characterisation of the FMP and degrees of FMP in extensions of bi-KG. Both of these make heavy use of the understanding of $\mathbb{U}(\mathscr{G})$.

We begin with the description of extensions of bi-KG with the FMP. In general, a logic enjoys the FMP if it is generated by its finite algebras. In our specific case, this requirement narrows down to verifying that the simple algebras validating the logic can be generated by the finite simple algebras. Furthermore, given a simple algebra A, we know the shape of the finite simple algebras that can generate A – these are what we call m-compressions of A, where m is a natural number. While the precise definition of an m-compression of A is quite technical, a quick intuition is that it is the result of replacing infinite segments of A with finite parts of size at least m.

These observations lead to the following result.

Theorem 6. An extension L of bi-KG has the FMP if and only if for each of its simple algebras A and each natural number m, there exists an m-compression of A satisfying L.

By applying this result to particular extensions, we get the the following corollaries.

Corollary 7. The logic bi-KG has the FMP.

Corollary 8. The logic generated by the bi-Heyting Rieger-Nishimura lattice lacks the FMP.

The latter is a notable difference with the intuitionistic case, where the logic generated by the Rieger-Nishimura lattice enjoys the FMP.

[1] There are actually two exceptions, but they play no important role.

5 Degrees of FMP in extensions of bi-KG

Using the work from the previous sections, we reach a full characterisation of degrees of FMP in extensions of bi-KG. Interestingly, we obtain a dichotomy-style theorem, in contrast with the KG case.

Theorem 9. In extensions of bi-KG, all possible degrees of FMP are 1 and $2^{\aleph_0}$.

In order to prove this statement, we follow the following strategy. Firstly, we observe that we already have witnesses of the degrees of FMP 1 and $2^{\aleph_0}$ – these are bi-KG and the logic generated by the bi-Heyting Rieger-Nishimura lattice respectively. This can be seen with the help of Corollary 7 and 8.

Secondly, we prove that if a given logic extending bi-KG has degree of FMP greater than 1, then its degree of FMP is $2^{\aleph_0}$. This is achieved through the explicit construction of continuum many logics with the same finite algebras. In particular, we build countably many algebras and generate continuum many logics by taking subsets of these algebras. The algebras are carefully selected in order to ensure that every subset of algebras generates a unique logic.

6 Directions for future work

A natural continuation of our work would be a characterisation of degrees of FMP for all extensions of bi-intuitionistic logic. We showed that the only know technique to construct finite degrees in intuitionistic logic does not work bi-intuitionistically and we believe that this hints at a possible dichotomy theorem for extensions of bi-intuitionistic logic.

Moreover, we find it interesting whether our ideas for bi-intuitionistic logic can be applied to other similar logical systems. For instance, we see temporal logic and intuitionistic modal logic as potential candidates.

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