From implicative reducts to Mundici's functor

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The connection between substructural logics and residuated lattices is one of the most relevant results of algebraic logic. Indeed, it establishes a framework where different systems, or equivalently, classes of structures, can be both compared and studied uniformly.

Among the most well-known connections among different structures in this framework surely stands Mundici's theorem, which establishes a categorical equivalence between the algebraic category of MV-algebras and lattice-ordered abelian groups (abelian ℓ-groups in what follows) with strong order unit (an archimedean element with respect to the lattice order), with unit preserving homomorphisms. This equivalence, connecting the equivalent algebraic semantics of infinite-valued Łukasiewicz logic (i.e., MV-algebras) with ordered groups, has been deeply investigated and also extended to more general structures. In particular, Dvurečenskij first extended Mundici's approach to the case where the monoidal operation involved is not commutative, showing a categorical equivalence between ℓ-groups with strong unit and *pseudo* MV-algebras.

Alternative algebraic approaches to Mundici's functor have been proposed by other authors. In particular, Galatos and Tsinakis in [5] extended both Mundici and Dvurčenskij's result to the unbounded and non-commutative setting of *generalized* MV-algebras, using a truncation construction based on the work of Bosbach on *cone algebras* [2, 3]. The connection between ℓ-groups and other relevant structures in algebraic logic is also deeply explored in [1].

In the present contribution we re-elaborate Rump's work [6], which is again inspired by Bosbach's ideas [2],[3],[4], but focuses on structures with only one implication and a constant (whereas Bosbach's cone algebras, used also in [5], have two implications). The key idea is to characterize which structures in this reduced signature embed in an ℓ-group. We find conditions that are different (albeit equivalent) to the ones found by Rump, and moreover we extend some of Rump's constructions to categorical equivalences of the algebraic categories involved.

Let us now give more details. The construction starts from structures having only one binary operation, which interprets some form of implication. We call *unital magma* a structure $\langle M, \to, 1\rangle$, satisfying the following equations and quasi-equation:

(M1) $x \to x \approx 1$;

(M2) $1 \to x \approx x$;

(M3) $x \to 1 \approx 1$;

(M4) $(x \to y \approx y \to x) \Rightarrow x \approx y$.

In particular, we call a unital magma a *H-algebra* if it satisfies two more equations:

(H) $(x \to y) \to (x \to z) \approx (y \to x) \to (y \to z)$;

(K) $x \to (x \to y) \approx 1$.

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With the properties (H) and (K) one can define a partial order on the structures in the usual way: $x \leq y$ iff $x \to y = 1$.

H-algebras turn out to be the appropriate framework to identify the fundamental properties that are satisfied by an implicative reduct of an ℓ-group, and thus they are the starting point for the construction. In particular, given any H-algebra $\mathbf{A}$, one first constructs the free generated monoid from $\mathbf{A}$, and then suitably considers a particular quotient in order to get a right-cancellative left-complemented (i.e., left-residuated) monoid, let us call it $\mathbf{L_A}$. The starting algebra $\mathbf{A}$ embeds (with respect to its reduced signature) in $\mathbf{L_A}$. The idea is now to embed $\mathbf{L_A}$ into the negative cone of an ℓ-group. Thus, the next step is to construct a group $\mathbf{G_{L_A}}$ from the previously obtained left-complemented monoid, with a construction that is similar to Øre's group of fractions. That is, considering $\mathbf{L_A}$, one can define the following equivalence relation on $L_A \times L_A$ to construct $G_{\mathbf{L_A}}$:

$$(a, b) \equiv (c, d) \text{ if and only if there exists } u, v \in L_A \text{ such that } ua = vc \text{ and } ub = vd.$$

Given particular properties of $\mathbf{L_A}$, $G_{\mathbf{L_A}}$ can be equipped with the operations of a partially ordered group. Then it follows from well-known results that cancellativity of $\mathbf{L_A}$ is a sufficient and necessary condition for $\mathbf{L_A}$ to be embedded as a partially ordered monoid into the negative cone of an ℓ-group. In order to obtain en embedding with respect to $\to$ (where in the ℓ-group, $x \to y = yx^{-1} \wedge 1$), one needs to require a condition called *regularity*, that can be expressed in terms of the implication $\to$. Thanks to regularity, the generated group $\mathbf{G_{L_A}}$ is in particular an ℓ-group. If one also requires the starting algebra $\mathbf{A}$ to be *full*, that is, for all $b, c \in A$, if $b \leq c$, there exists $a \in A$ such that $a \to b = c$, one gets an isomorphism between $\mathbf{L_A}$ and the negative cone of $\mathbf{G_{L_A}}$ in the reduced signature. Fullness and regularity turn out to be necessary and sufficient conditions for a cancellative, left complemented monoid to be isomorphic with the negative cone of an ℓ-group (seen as a left complemented monoid). Moreover, this characterization can be extended to a categorical equivalence. Indeed we can show that the algebraic categories of negative cones of ℓ-groups and of full, regular, cancellative, left-complemented monoids are equivalent.

In order to recover Mundici's theorem, we focus our attention on bounded H-algebras. In order to characterize embeddings for bounded H-algebras, we show the following lemma:

Lemma 1. *A bounded H-algebra* $\mathbf{A}$ *is* $\to$*-isomorphic with the interval* $[u, 1]$ *of a given* ℓ*-group with strong unit* u*, if and only if*

- *for all* $a, b \in A$*, if there exists* $v \in \mathbf{L_A}$*,* $v \leq a, b$*, such that* $a \to v = b \to v$*, then* $a = b$*;*
- A *is full and regular.*

As Rump's observes in [6], if an H-algebra satisfies Tanaka's equation, then $\mathbf{L_A}$ is commutative as a monoid, it has a lattice order, and moreover, we show that it satisfies the conditions of the previous lemma. Thanks to this observation, we gain that given any $\mathbf{A}$ full, regular and bounded H-algebra satisfying Tanaka's equation, then $\mathbf{A}$ is $\to$-isomorphic to the interval $[u, 1]$ of the negative cone of an ℓ-group. The latter extends to Mundici's result.
In order to deal with the non-commutative case, a little more work is required to show that the two implications can be recovered with this construction. Nonetheless, we can prove that, with some further technicalities, the same construction works. In the process we get another categorical equivalence of some interest and as another particular case, Dvurečenskij's result.

References

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