Connexive implication in substructural logics

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Connexive Logic is a stream of research devoted to formalize conditionals expressing coherence/connection requirements between their antecedent and consequent. The current interest in these logics relies on their capability of formalizing indicative natural language conditionals (see [1]), counterfactuals (see [4]), and some species of physical and "causal" implications (see [3]).

We say a logic $\mathcal{L}$ is connexive provided that it has a negation $\neg$ and an implication $\to$ satisfying *Aristotle's Theses*:

$$\neg(\alpha \to \neg\alpha) \tag{AT1}$$

$$\neg(\neg\alpha \to \alpha) \tag{AT2}$$

e.g., that no formula implies or is implied by its own negation; *Boethius' Theses*:

$$(\alpha \to \beta) \to \neg(\alpha \to \neg\beta) \tag{BT1}$$

$$(\alpha \to \neg\beta) \to \neg(\alpha \to \beta) \tag{BT2}$$

e.g., that if α implies β (respectively, $\neg\beta$), then it is not the case that α implies β (respectively, $\neg\beta$) as well; and lastly, and crucially, the stipulation that $\to$ be non-symmetric, as to properly distinguish it from bi-implication. Apparently, these theses are falsified by classical logic whenever implications with false antecedents are considered.

Over the past years the research on connexive logic has been focused on defining new deductive systems satisfying connexive principles. However, to the best of our knowledge, the literature does not offer a systematic attempt to verify to what extent familiar systems of non-classical logic, e.g. substructural logics, admit (definable) connexive implications. At least not until recently where, in the work of Fazio, Ledda, and Paoli, it is shown that intuitionistic logic is deductively equivalent to their so-called *Connexive Heyting logic*. From the semanitcal perspective, they show that the variety HA of Heyting algebras is term-equivalent to a class of *Connexive Heyting algebras*. In particular, they show that in HA, the operation $\Rightarrow$ defined via

$$x \Rightarrow y := (x \to y) \wedge (y \to \neg\neg x),$$

where $\to$ is Heyting implication and $\neg x := x \to 0$ is Heyting negation, is generally non-symmetric and, in conjunction with Heyting negation, satisfies (the equational renderings of) laws for a connexive implication, i.e., Aristotle's and Boethius' theses.

Contributing to this line of research, we consider a broader class of substructural logics *vis-à-vis* their semantic lens in residuated structures. That is, we investigate those (sub)classes of commutative pointed residuated lattices, i.e., $\mathsf{FL_e}$-algebras, for which $\Rightarrow$, and similarly related operations, satisfy such connexive principles. We demonstrate that these properties are intimately related-to, and in many cases equivalent-to, having the equational Glivenko property hold relative to Boolean algebras (see [2] for more on the Glivenko property).

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In particular, given an $\mathsf{FL_e}$-algebra $\mathbf{A}$ and a operation $\Rightarrow : A \times A \to A$, we say $(\mathbf{A}, \Rightarrow)$ is *weakly connexive* if the following identities are satisfied:

$$1 \leq \neg(x \Rightarrow \neg x) \tag{at1}$$
$$1 \leq \neg(\neg x \Rightarrow x) \tag{at2}$$
$$1 \leq (x \Rightarrow y) \Rightarrow \neg(x \Rightarrow \neg y) \tag{bt1}$$
$$1 \leq (x \Rightarrow \neg y) \Rightarrow \neg(x \Rightarrow y) \tag{bt2}$$

and we say $\mathbf{A}$ is *connexive* if furthermore $\Rightarrow$ is non-symmetric. We prove the following:

Theorem 1. *Let C be the class of $\mathsf{FL_e}$-algebras satisfying the equation (bt1) (Boethius' thesis) for the connective $x \Rightarrow y := (x \to y) \wedge (y \to \neg\neg x)$, where $\neg x := x \to 0$. Then the following hold:*

1. *C is connexive, i.e., $(\mathbf{A}, \Rightarrow)$ is weakly connexive for every member $\mathbf{A}$ of C and $\Rightarrow$ is not generally symmetric in C;*
2. *C is exactly $\mathbf{G}_{\mathsf{FL_e}}(\mathsf{BA})$, the largest variety of $\mathsf{FL_e}$-algebras for which the equational Glivenko property holds relative to Boolean algebras.*

We also investigate those subvarieties of $\mathsf{FL_e}$ that are integral and/or where 0 is the least element along with a broader class of candidate connexive arrows. In particular, for the class $\mathsf{FL_{ew}}$ of 0-bounded integral $\mathsf{FL_e}$-algebras, we obtain the following:

Theorem 2. *Let $\mathbf{A}$ be an $\mathsf{FL_{ew}}$-algebra and define the operations $\Rightarrow_\circ$ and $\Rightarrow_\wedge$ on $\mathbf{A}$ via:*

$$x \Rightarrow_\circ y := (x \to y) \cdot (y \to \neg\neg x)$$
$$x \Rightarrow_\wedge y := (x \to y) \wedge (y \to \neg\neg x)$$

and note that the interval $[\Rightarrow_\circ, \Rightarrow_\wedge]$ of binary operation (under the usual ordering) is non-empty. Then the following are equivalent:

1. *$\mathbf{A}$ is a member of $\mathbf{G}_{\mathsf{FL_{ew}}}(\mathsf{BA})$, the largest variety of integral 0-bounded $\mathsf{FL_e}$-algebras for which the equational Glivenko property holds relative to Boolean algebras.*
2. *For all $\Rightarrow \in [\Rightarrow_\circ, \Rightarrow_\wedge]$, $(\mathbf{A}, \Rightarrow)$ is weakly connexive.*
3. *There exists $\Rightarrow \in [\Rightarrow_\circ, \Rightarrow_\wedge]$ such that $(\mathbf{A}, \Rightarrow) \models$ (at1) (Aristotle's thesis).*

References

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