Probability via Łukasiewicz logic: a multi-type semantic and proof theoretical account

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Providing good proof systems for probabilistic logics is a long standing problem in proof theory and logics for uncertainty. This work is a part of larger research project aimed at providing good proof systems for probabilistic logics and other logics of uncertainty in a uniform and modular way. In this project, we use a generalization of display calculi introduced by Belnap. This choice is motivated by the following two reasons. Firstly, display calculi are by design modular, insofar they implement a neat division of labour between logical rules (introducing the connectives and relying on their minimal order-theoretic properties) and so-called structural rules (capturing the specific features of the logic under consideration). Secondly, they provide a framework in which cut-elimination, a crucial property of proof systems, can be proved in a principled way as an application of a general meta-theorem.

Logics for reasoning about probability have been extensively studied. In 1990, [4] introduces a logic to reason about probabilities and its Hilbert style calculus that contains three types of axioms and rules: the ones that govern the arithmetical part, i.e., the reasoning about inequalities; the ones that axiomatise probabilities; and the rules and axioms of classical propositional logic. In [13] and later in [12], probabilities are axiomatized via fuzzy logics, in a language with two sorts: a sort for expressing Boolean statements and a sort for statements about probabilities. This two-layer approach for probabilistic logics is further developed in [6, 5, 7]. Finally, in 2020, [1] utilises a two-layered modal logic to formalise reasoning about probabilities. The proposed calculus consists of three parts: the rules and axioms of the logic of events (i.e. classical logic) or 'inner logic'; the 'outer logic' that formalises reasoning with probabilities; and finally, the modalities that transform events into probabilistic statements.

The main difficulties in applying the theory of display calculi to the probability logics lies in the handling of the operators $+$ and $-$ (i.e. the truncated sum and difference, respectively) and their interaction with the probability operator P in well-known axiomatization of probability. Here we rely on an ongoing work, where we introduce a generalization of standard display calculi to capture Łukasewicz logic and, in particular, to deal with the axiom

$$((A \to B) \to B) \to (A \vee B) \tag{1}$$

which can be equivalently written as

$$((A - B) + B) \to (A \vee B)$$

and which is closely connected to the probability axiom

$$((P(A) - P(A \wedge B)) + P(B)) \to P(A \vee B).$$

Łukasiewicz logic is one of the most well-know and thoroughly studied mathematical fuzzy logics (see [14] for an overview of proof theoretic literature on mathematical fuzzy logics).

Nonetheless, the distinctive axiom of Łukasiewicz logic 1 is not **analytic-inductive** [11] (not even canonical) and it represents the main obstacle to a uniform and modular proof-theoretic treatment. Pivoting on an algebraic analysis of Łukasiewicz logic, we introduce a refinement of the general theory of display sequent calculi and algorithmic rule generation (as developed for instance in [8] and [11], respectively) aiming at overcoming this problem. In particular, we rely on the fact that Łukasiewicz operators are not only normal operators, but also regular operators in the following sense (in [10] and [9] such operators are called 'double quasioperators'):

normal binary diamond	normal binary box
$A \odot \mathbf{0} \;=\; \mathbf{0} \;=\; \mathbf{0} \odot A$ $(A \vee B) \odot C = (A \odot C) \vee (B \odot C)$ $C \odot (A \vee B) = (C \odot A) \vee (C \odot B)$	$A \oplus \mathbf{1} \;=\; \mathbf{1} \;=\; \mathbf{1} \oplus A$ $(A \wedge B) \oplus C = (A \oplus C) \wedge (B \oplus C)$ $C \oplus (A \wedge B) = (C \oplus A) \wedge (C \oplus B)$
$A \ominus \mathbf{1} \;=\; \mathbf{0} \;=\; \mathbf{0} \ominus A$ $(A \vee B) \ominus C = (A \ominus C) \vee (B \ominus C)$ $C \ominus (A \wedge B) = (C \ominus A) \vee (C \ominus B)$	$A \rightarrow \mathbf{1} \;=\; \mathbf{1} \;=\; \mathbf{0} \rightarrow A$ $(A \vee B) \rightarrow C = (A \rightarrow C) \wedge (B \rightarrow C)$ $C \rightarrow (A \wedge B) = (C \rightarrow A) \wedge (C \rightarrow B)$

regular binary diamond	regular binary box
$(A \vee B) \oplus C = (A \oplus C) \vee (B \oplus C)$ $C \oplus (A \vee B) = (C \oplus A) \vee (C \oplus B)$	$(A \wedge B) \odot C = (A \odot C) \wedge (B \odot C)$ $C \odot (A \wedge B) = (C \odot A) \wedge (C \odot B)$
$(A \wedge B) \rightarrow C = (A \rightarrow C) \vee (B \rightarrow C)$ $C \rightarrow (A \vee B) = (C \rightarrow A) \vee (C \rightarrow B)$	$(A \wedge B) \ominus C = (A \ominus C) \wedge (B \ominus C)$ $C \ominus (A \vee B) = (C \ominus A) \wedge (C \ominus B)$

Exploiting the previous observation, we introduce a language expansion where the different "personalities" (normal versus regular) of the operators are fully-fledged and, in turn, it becomes possible to introduce a sequent calculus with the so-called **relativized display property** (namely, every structure occurring in a derivable sequent is displayable). Moreover, all the logical introduction rules are standard and reflect the minimal order-theoretic properties of the operators, while the specific features of the logic are captured by so-called structural rules, so maintaining a neat division of labour that guarantees a modular treatment. At last, all the structural rules are automatically generated via (a specialisation of) the algorithm ALBA (to regular operators). Showing that the calculus enjoys (canonical) cut elimination is future work.

Below we expand on the treatment of the probability operator. The key idea is that the non-normal operators (like the conditional binary operator of conditional logics or the monotone unary modalities in non-normal modal logics) can be decomposed into the composition of normal modal operators [3]. In this work, we use a similar approach to deal with the probability operator P.

Let $\mathcal{B}$ be any set and $\mathcal{P}(\mathcal{B})$ be its power-set. Let $P : \mathcal{P}(\mathcal{B}) \rightarrow [0, 1]$ be a probability function on it. Let $R_{\in}, R_{\notin} \subseteq \mathcal{P}(\mathcal{B}) \times \mathcal{B}$ be defined as follows. For any $a \in \mathcal{B}$, $A \in \mathcal{P}(\mathcal{B})$,

$$AR_{\in}a \quad \text{iff} \quad a \in A \quad \text{and} \quad AR_{\notin}a \quad \text{iff} \quad a \notin A.$$

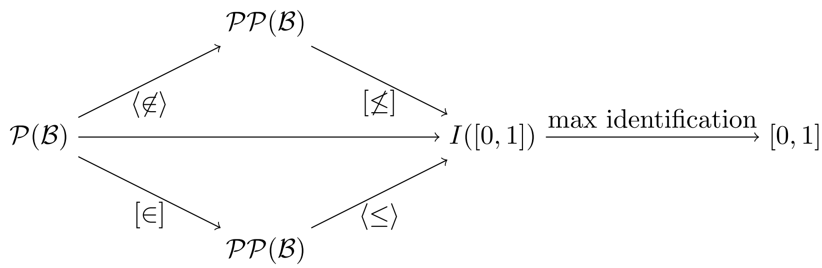

Figure 1: Decomposition of the probability operator P using normal operators

Let $R_{\leq}, R_{\not\leq} \subseteq [0,1] \times \mathcal{P}(\mathcal{B})$ be defined as follows. For any $\alpha \in [0,1]$, $A \in \mathcal{P}(\mathcal{B})$,

$$\alpha R_{\leq} A \quad \text{iff} \quad \alpha \leq P(A) \quad \text{and} \quad \alpha R_{\not\leq} A \quad \text{iff} \quad \alpha \not\leq P(A).$$

Let $A \subseteq \mathcal{B}$, and $U \subseteq \mathcal{P}(\mathcal{B})$ be any subsets of $\mathcal{B}$ and $\mathcal{P}(\mathcal{B})$ respectively. Let $[\in](A) = [R_{\in}](A)$, $\langle\notin\rangle(A) = \langle R_{\notin}\rangle(A)$, $\langle\leq\rangle(U) = \langle R_{\leq}\rangle(U)$, and $[\not\leq](U) = [R_{\not\leq}](U)$. Then, we have

Lemma 1. *For any $A \subseteq \mathcal{B}$, and $U \subseteq \mathcal{P}(\mathcal{B})$,*

1. $[\in](A) = A^{\downarrow}$.
2. $\langle\notin\rangle(A) = (A^{\uparrow})^c$.
3. $\langle\leq\rangle(U) = [0, \max\{P(A) \mid A \in U\}]$.
4. $[\not\leq](U) = [0, \min\{P(A) \mid A \in U^c\}]$.

The following corollary follows immediately from the Lemma.

Corollary 2. *For any $A \subseteq \mathcal{B}$, $P(A) = \max(\langle\leq\rangle[\in](A)) = \max([\not\leq]\langle\notin\rangle(A))$.*

Thus, under the identification of an interval with its largest element above, the corollary shows that the probability operator P can be decomposed into the combination of normal operators $\langle\leq\rangle$, $[\in]$, $[\not\leq]$, and $\langle\notin\rangle$ in two ways. This decomposition allows us to write the probability axioms in the language of Łukasewicz logic expanded with the above modal operators. Therefore, the axioms of probability logic can be expressed in the above multi-type normal modal logic.

Finally, the difficult (non-analytic) axiom in the probability theory is the inclusion-exclusion axiom. This axiom is very similar to the peculiar axiom of Łukasewicz logic discussed earlier (with the addition of the operator P). In this talk we will expand on the work in progress aiming at introducing a properly displayable multi-type calculus for probability logic. Showing that the calculus enjoys (canonical) cut elimination is future work.

We believe that these techniques would allow us to deal with other (non-classical) logics of uncertainty such as the logics for probabilities and belief functions over Belnap-Dunn logic introduced in [2].

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