Strictly join irreducible varieties of residuated lattices

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Substructural logics constitute a large class of logical systems algebraizable in the sense of Blok-Pigozzi, where the semantical characterization of provability of the Lindenbaum-Tarski algebraization extends to a characterization of logical deducibility via the algebraic equational consequence (see [4] for a detailed investigation). Substructural logics encompass classical logic, intermediate logics, fuzzy logics, relevance logics and many other systems, all seen as logical extensions of the Full Lambek calculus $\mathcal{FL}$. As a consequence of algebraizability, all extensions of $\mathcal{FL}$ are also algebraizable, and the lattice of axiomatic extensions is dually isomorphic to the subvariety lattice of the algebraic semantics, given by the variety of FL-algebras. In this work we are interested in the positive fragment of $\mathcal{FL}$ (the system obtained by removing the constant 0, and consequently negation, from the language), $\mathcal{FL}^+$, whose corresponding algebraic semantics is given by the variety of residuated lattices RL.

Our investigation will be carried on in the algebraic framework, and goes in the direction of gaining a better understanding of the lattice of subvarieties of residuated lattices (thus, equivalently, the lattice of axiomatic extensions of the corresponding logics). In particular we study properties, and in some relevant cases we find characterizations, of those varieties that in the lattice of subvarieties are join irreducible or strictly join irreducible. Kihara and Ono showed that, in presence of integrality and commutativity, join irreducibility of a variety is characterized by both a logical property, Halldén completeness, and by an algebraic property of the generating algebras. A substructural logic $\mathcal{L}$ has the **disjunction property** if whenever $\varphi \vee \psi$ is a theorem of $\mathcal{L}$, in symbols $\mathcal{L} \vdash \varphi \vee \psi$, then either $\mathcal{L} \vdash \varphi$ or $\mathcal{L} \vdash \psi$. Likewise a commutative and integral residuated lattice $\mathbf{A}$ is **well-connected** if 1 is join irreducible, i.e. $a \vee b = 1$ implies either $a = 1$ or $b = 1$. A weaker property is Halldén completeness; a logic L is **Halldén complete** if it has the disjunction property w.r.t. to any pair of formulas that have no variables in common. Classical logic is Halldén complete but does not have the disjunction property, thus differentiating the two concepts. As shown in [5] these concepts are connected in commutative integral residuated lattices.

Theorem 0.1. *(Theorem 2.5 in [5]) For a variety* V *of commutative and integral residuated lattices the following are equivalent:*

1. $\mathcal{L}_{\mathsf{V}}$ *is Halldén complete;*
2. V *is join irreducible;*
3. $\mathsf{V} = \mathsf{V}(\mathbf{A})$ *for some well-connected algebra* $\mathbf{A}$.

How can we extend the definition of well-connected to the nonintegral case? The solution proposed in [5] (and later followed in [2]) is to define a residuated lattice $\mathbf{A}$ to be **well-connected** if 1 is **join prime** in $\mathbf{A}$, i.e. $a \vee b \geq 1$ implies $a \geq 1$ or $b \geq 1$.

We observe straight away that neither integrality nor commutativity are needed to prove that (3) implies (2).

Lemma 0.2. *Let* V *be a variety of residuated lattices; if* $\mathsf{V} = \mathbf{V}(\mathbf{A})$ *for some well-connected algebra* $\mathbf{A} \in \mathsf{V}$ *then* V *is join irreducible.*

The other implications in the general case however do not hold; an analysis of the Kihara-Ono construction reveals at once that there are two critical points. If V is a variety of commutative residuated lattices then:

- every subdirectly irreducible algebra in V is well-connected ([5], Lemma 2.2);
- if W, Z are subvarieties of V axiomatized (relative to V) by $p \geq 1$ and $q \geq 1$ (and we make sure that p and q have no variables in common), then $\mathsf{W} \vee \mathsf{Z}$ is axiomatized relative to V by $p \vee q \geq 1$ ([5], Lemma 2.1).

Both statements are false if we remove commutativity; for the first it is easy to find a finite and integral residuated lattice that is simple but not well-connected (for instance the example below Lemma 3.60 in [4]), while the second fails fore more general reasons discussed at length in [3].

However it is possible to prove a similar result for non-integral, non-commutative subvarieties of RL, characterizing join irreducibility in a large class of residuated lattices, that include for instance all normal varieties, representable varieties, and ℓ-groups. To do so we will adapt to our purpose part of the theory developed in [3] about satisfaction of formulas generated by iterated conjugates.

We define a set $B^n(x, y)$ of equations in two variables x, y for all $n \in \mathbb{N}$ in the following way; let Γ^n be the set of iterated conjugates of *length* n (i.e. a composition of n left and right conjugates) over the appropriate language, with $\Gamma^0 = \{l_1\}$ (for a more general definition, here not needed, see [3], page 229). For all $n \in \mathbb{N}$

$$B^n(x, y) = \{\gamma_1(x) \vee \gamma_2(y) \approx 1 \, : \, \gamma_1, \gamma_2 \in \Gamma^n\}.$$

Let $\mathbf{A}$ be a residuated lattice and $a, b \in A$; we say that $\mathbf{A}$ **satisfies** $B^n(a, b)$, in symbols $\mathbf{A} \vDash B^n(a, b)$ if $\mathbf{A}, a, b \vDash B^n(x, y)$. i.e. $\gamma_1(a) \vee \gamma_2(b) = 1$ for all $\gamma_1, \gamma_2 \in \Gamma^n(\mathbf{A})$. We say that $\mathbf{A}$ satisfies $(G_{n,k})$ if for all $a, b \in A$, if $\mathbf{A} \vDash B^n(a, b)$, then $\mathbf{A} \vDash B^k(a, b)$.

This lemma is implicit in [3].

Lemma 0.3. *Let* V *be a variety of residuated lattices and let* $p(x_1, \ldots, x_n) \geq 1$, $q(y_1, \ldots, y_m) \geq 1$ *be two inequalities not holding in* V. *If* W *and* Z *are the subvarieties axiomatized by* $p \wedge 1 \approx 1$ *and* $q \wedge 1 \approx 1$ *respectively, then* $\mathsf{W} \vee \mathsf{Z}$ *is axiomatized by the set* $B(p, q) = \bigcup_{n \in \mathbb{N}} B^n(p, q)$. *Moreover if* V *satisfies* $(G_{l,l+1})$ *for some* $l \in \mathbb{N}$ *then* $\mathsf{W} \vee \mathsf{Z}$ *is axiomatized by the finite set* $B^l(p, q)$.

We say that a residuated lattice $\mathbf{A}$ is $\mathbf{\Gamma^n}$**-connected** if for all $a, b \in A$, if $\gamma_1(a) \vee \gamma_2(b) = 1$ for all $\gamma_1, \gamma_2 \in \Gamma_n(\mathbf{A})$, then either $a \geq 1$ or $b \geq 1$.

Lemma 0.4. *Let* V *be a variety of residuated lattices that satisfies* $(G_{n,n+1})$. *Then every subdirectly irreducible algebra in* V *is* Γ^n*-connected.*

Finally let's complete the connection with logic. Let $\mathcal{L}$ be a substructural logic over $\mathcal{FL}^+$; given any two axiomatic extensions $\mathcal{L}_1$ and $\mathcal{L}_2$ axiomatized by formulas ϕ and ψ respectively, for any n Lemma 0.3 implicitly gives a set of formulas $B^n_{\mathcal{L}}(\phi, \psi)$ such that $B_{\mathcal{L}}(\phi, \psi) = \bigcup_{n \in \mathbb{N}} B^n_{\mathcal{L}}(\phi, \psi)$ axiomatizes the intersection $\mathcal{L}_1 \cap \mathcal{L}_2$, corresponding to the join of the varieties $\mathsf{V}_{\mathcal{L}_1} \vee \mathsf{V}_{\mathcal{L}_2}$. We say that $\mathcal{L}$ is $\mathbf{\Gamma^n}$**-complete** if for all formulas ϕ and ψ which have no variables in common, if $\mathcal{L} \vdash B^n_{\mathcal{L}}(\phi, \psi)$ then either $\mathcal{L} \vdash \phi$ or $\mathcal{L} \vdash \psi$.

Theorem 0.5. *Let* V *be a variety of residuated lattices satisfying* $(G_{n,n+1})$ *for some* $n \in \mathbb{N}$*; then the following are equivalent.*

1. $\mathcal{L}_{\mathsf{V}}$ *is* Γ^n*-complete;*
2. V *is join irreducible;*
3. $\mathsf{V} = \mathbf{V}(\mathbf{A})$ *for some* Γ^n*-connected algebra* $\mathbf{A}$*.*

Next we point out a corollary of Lemma 0.4 and Theorem 0.5.

Corollary 0.6. *Let* V *be a variety of residuated lattices satisfying* $(G_{n,n+1})$ *for some* $n \in \mathbb{N}$*. If there is a subdirectly irreducible algebra* $\mathbf{A}$ *with* $\mathsf{V} = \mathbf{V}(\mathbf{A})$*, then* V *is join irreducible.*

This is the analogue of Lemma 2.6(2) in [5] and the authors asked if it was possible to invert it; it turns out that our (more general) version is indeed invertible, thus answering their question as well.

Theorem 0.7. *Let* V *be a variety of residuated lattices that satisfies* $(G_{n,n+1})$ *for some* $n \in \mathbb{N}$*; if* V *is join irreducible, then there is a subdirectly irreducible algebra* $\mathbf{B} \in \mathsf{V}$ *such that* $\mathbf{V}(\mathbf{B}) = \mathsf{V}$*.*

We observe that the above results can (and have been) used to characterize all strictly join irreducible varieties of basic hoops and all linear varieties of basic hoops. Finally we point out that all the material covered in this abstract has appeared in [1].

References

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