Structural completeness and lattice of extensions in many-valued logics with rational constants

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1 Introduction

The logics **RŁ**, **RP**, and **RG** are obtained by expanding Łukasiewicz logic **Ł**, product logic **P**, and Gödel logic **G** with rational constants $\{\mathbf{c}_q : q \in [0,1] \cap \mathbb{Q}\}$ and adding the bookkeeping axioms: For every $p, q \in [0,1] \cap \mathbb{Q}$,

$$\mathbf{c}_p \cdot \mathbf{c}_q \leftrightarrow \mathbf{c}_{p*q} \qquad (\mathbf{c}_p \to \mathbf{c}_q) \leftrightarrow \mathbf{c}_{p \Rightarrow q} \qquad \mathbf{c}_0 \leftrightarrow \bot \qquad \mathbf{c}_1$$

where $*$ is the Łukasiewicz, Product, and Gödel (minimum) t-norm and $\Rightarrow$ is the Łukasiewicz, Product, and Gödel standard residuated implication in each case.

The history of these logics goes back to the pioneering works of Goguen [13] and Pavelka [20, 21, 22]. Expanding the language with constants can be viewed as taking advantage of the rich algebraic setting to gain more expressivity; see, e.g., [2, 4, 9, 10, 15, 23, 24]. In this talk, we study the lattices of extensions and structural completeness of these three expansions, obtaining results that stand in contrast to the known situation in **Ł**, **P**, and **G**.

A *rule* is an expression of the form $\Gamma \rhd \varphi$, where $\Gamma \cup \{\varphi\} \subseteq Fm$ is a finite set. A rule $\Gamma \rhd \varphi$ is said to be *derivable* in a logic $\vdash$ when $\Gamma \vdash \varphi$. It is *admissible* in $\vdash$ when for every substitution σ on Fm,

$$\text{if } \emptyset \vdash \sigma(\gamma) \text{ for all } \gamma \in \Gamma, \text{ then } \emptyset \vdash \sigma(\varphi).$$

In other words, a rule is admissible in $\vdash$ when its addition to $\vdash$ does not produce any new theorem. Clearly, every rule that is derivable in $\vdash$ is also admissible in $\vdash$. If the converse holds, $\vdash$ is said to be *structurally complete* (SC). Logics whose extensions are all structurally complete have been called *hereditarily structurally complete* (HSC).

During the last two decades, research in structural completeness has turned also to the family of fuzzy logics. While **G** and **P** are hereditarily structurally complete [8, 5], **Ł** is structurally incomplete [7] and a base for its admissible rules was exhibited by Jeřábek [17], see also [16, 18]. Admissibility in extensions of **Ł** was investigated in [11, 12].

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Łukasiewicz logic **Ł**, product logic **P**, and Gödel logic **G** can be obtained as axiomatic extensions of Hájek's basic logic **BL** [14], even if they were defined independently prior to the definition of **BL**. All logics in the **BL** family are algebraizable in the sense of Blok and Pigozzi [3]: the equivalent algebraic semantics of the three logics are the varieties of MV-algebras, product algebras and Gödel algebras, respectively. In fact, from algebraizablity we obtain a dual lattice isomorphism from the lattice of finitary extensions of each logic $L_E(\vdash)$ into the lattice of quasivarieties of each equivalent variety semantics $L_Q(\mathbb{V})$. Moreover, if we restrict this isomorphism to the lattice of axiomatic extensions $L_{AE}(\vdash)$ we get an isomorphism $L_{AE}(\vdash) \cong L_V(\mathbb{V})$, where $L_V(\mathbb{V})$ denotes the lattice of all subvarieties of $\mathbb{V}$.
Komori in [19] characterizes $L_{AE}(\mathbf{Ł})$ which forms an infinite non totally ordered denumerable pseudo Boolean algebra. The lattice of all extensions $L_E(\mathbf{Ł})$ is as complicated as it can be, since the class of all MV-algebras $\mathbb{MV}$ is Q-universal [1]. That is, for every quasivariety $\mathbb{K}$ of finite type $L_Q(\mathbb{K})$ is a homomorphic image of a sublattice of $L_Q(\mathbb{MV})$.
The lattice of all axiomatic extensions of **P** is just the three element chain where the only consistent proper axiomatic extension of **P** is classical logic. Since **P** is hereditary structurally complete $L_E(\mathbf{P}) = L_{AE}(\mathbf{P})$ [6].
Finally, every axiomatic consistent extension of **G** is a finite valued Gödel logic and $L_{AE}(\mathbf{G}) \cong \omega + 1$. Since **G** is hereditary structurally complete $L_E(\mathbf{G}) \cong \omega + 1$ (see [8]).
Structural completeness and the structure of the lattice of axiomatic extensions and the lattice of extensions need not to be preserved when expanding with rational constants, while algebraizability is preserved:

(i) **RŁ** is an algebraizabe conservative expansion of **Ł** and the variety of all rational MV-algebras $\mathbb{RMV}$ is its equivalent variety semantics.

(ii) **RP** is an algebraizable conservative expansion of **P** and the variety of all rational product algebras $\mathbb{RP}$ is its equivalent variety semantics.

(iii) **RG** is an algebraizable conservative expansion of **G** and the variety of all rational Gödel algebras $\mathbb{RG}$ is its equivalent variety semantics.

We recall that a *rational MV-algebra* , *rational product algebra* and *rational Gödel algebra* is an algebra **A** in the language $\mathcal{L} = \{\wedge, \vee, \cdot, \to, \bot, \top\} \cup \{\mathbf{c}_q : q \in [0,1] \cap \mathbb{Q}\}$ such that the $\{\wedge, \vee, \cdot, \to, \bot, \top\}$-reduct is an MV-algebra, Product algebra and Gödel algebra respectively and it satisfies the following bookkeeping equations: For every $p, q \in [0,1] \cap \mathbb{Q}$,

$$\mathbf{c}_p \cdot \mathbf{c}_q \approx \mathbf{c}_{p*q} \qquad (\mathbf{c}_p \to \mathbf{c}_q) \approx \mathbf{c}_{p\Rightarrow q} \qquad \mathbf{c}_0 \approx \bot \qquad \mathbf{c}_0 \approx \top$$

2 Main results

2.1 Rational Łukasiewicz logic

For the case of Łukasiewicz adding rational constants trivializes the lattice of extensions:

Theorem 2.1. *RŁ has no proper consistent extensions, hence RŁ is hereditary structurally complete.*

$$L_E(\boldsymbol{R\L}) = L_{AE}(\boldsymbol{R\L}) \cong 2$$

2.2 Rational Product logic

In the case of product logic adding rational constants does not have a significant change in the lattice of axiomatic extensions

Theorem 2.2. **RP** *has two proper consistent axiomatic extensions: namely PL and CL.*

- *PL is axiomatized by c_q for each (some) $q \in (0,1] \cap \mathbb{Q}$*
- *CL is axiomatized by c_q for each (some) $q \in (0,1] \cap \mathbb{Q}$ plus $\varphi \vee (\varphi \to \bot)$*

Corollary 2.3. *$L_{AE}(\mathbf{RP})$ is a four element chain.*

Notice that PL is equivalent to the original **P** and CL is equivalent to classical logic, hence when studying admissible rules we will only need to study admissble rules for **RP**.

Theorem 2.4. *Every proper extension of* **RP** *is structurally complete, but* **RP** *is not structurally complete. A base for the admissible rules of* **RP** *is given by the set of rules of the form*

$$\mathbf{c}_q \vee z \rhd z \qquad (\mathbf{c}_p \leftrightarrow x^n) \vee z \rhd z,$$

for each (equiv. some) $q \in (0,1) \cap \mathbb{Q}$ and each $p \in [0,1] \cap \mathbb{Q}$, $n \in \omega$ such that $\sqrt[n]{p}$ is irrational.

Finally, the biggest contrast expanding with rational constants is in the lattice of extensions. We can not obtain a nice description of the lattice $L_E(\mathbf{RP})$ because of the following result:

Theorem 2.5. *The variety $\mathbb{RP}$ is Q-universal.*

2.3 Rational Gödel Logic

Expanding Gödel logic with rational constants have a significant efect in the lattice of axiomatic extensions. In fact next result shows that we go from a numerable chain to an uncountable chain.

Theorem 2.6. *Every consistent axiomatic extension of* **RG** *is of the form*
$\mathbf{RG}_r := \mathbf{RG} + \{\mathbf{c}_q : q \in [r,1] \cap \mathbb{Q}\}$ *for some* $r \in (0,1]$,
$\mathbf{RG}_p^\omega := \mathbf{RG} + \{\mathbf{c}_q : q \in (p,1] \cap \mathbb{Q}\}$ *for some rational* $p \in [0,1)$ *or*
$\mathbf{RG}_p^n := \mathbf{RG}_p^\omega + \bigvee_{0 \leqslant i < j \leqslant n+2} (\mathbf{c}_p \vee x_i) \leftrightarrow (\mathbf{c}_p \vee x_j)$ *for some rational* $p \in [0,1)$ *and* $n \in \omega$.
Moreover, $L_{AE}(\mathbf{RG})$ is an uncountable chain dually isomorphic to the poset obtained adding a new bottom element to the Dedekind–MacNeille completion of the lexicographic order of $([0,1) \cap \mathbb{Q}) \times (\omega + 1)$.

Observe that $\mathbf{RG}_1 = \mathbf{RG}$ and that $\mathbf{RG}_0^\omega$ is equivalent to **G** and $\mathbf{RG}_0^n$ is equivalent to the $(n+2)$-valued Gödel logic. Next result shows that none of the other extensions of **RG** is structurally complete.

Theorem 2.7. *The only consistent axiomatic extensions of* **RG** *structurally complete are* $\mathbf{RG}_0^\omega$ *and* $\mathbf{RG}_0^n$ *for each $n \in \omega$. Moreover, for all $r \in (0,1]$, $p \in [0,1) \cap \mathbb{Q}$, and $\gamma \in \omega + 1$:*

- *A base for the admissible rules of $\mathbf{RG}_r$ is given by the rules of the form $\mathbf{c}_q \vee z \rhd z$, for all $q \in [0,r) \cap \mathbb{Q}$;*
- *A base for the admissible rules of $\mathbf{RG}_p^\gamma$ is given by the rule $\mathbf{c}_p \vee z \rhd z$.*

If we denote by $\overline{\mathbf{RG}}_r$ the structural completion of $\mathbf{RG}_r$, then $\{\overline{\mathbf{RG}}_r : r \in (0,1]\}$ is an uncountable antichain in $L_E(\mathbf{RG})$. Conseqüently, $L_E(\mathbf{RG})$ seems not easy to describe since it contains an uncountable antichain and, by Theorem 2.6, it contains an uncountable chain. The question whether $\mathbb{RG}$ is Q-universal remains open.

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