Some Proof-theoretical aspects of non-associative, non-commutative multi-modal linear logic

EBEN BLAISDEL[1], MAX KANOVICH[2], STEPAN KUZNETSOV[3], ELAINE PIMENTEL[2]*, AND ANDRE SCEDROV[1]

[1] University College London, UK
{m.kanovich,e,pimentel}@ucl.ac.uk

[2] Steklov Mathematical Institute of RAS, Russia
stephan.kuznetsov@gmail.com

[3] University of Pennsylvania, USA
{scedrov@math, ebenb@sas}.upenn.edu

Abstract

Adding multi-modalities (called *subexponentials*) to linear logic enhances its power as a logical framework, which has been extensively used in the specification of *e.g.* proof systems and programming languages. Initially, subexponentials allowed for classical, linear, affine or relevant behaviors. Recently, this framework was enhanced so to allow for commutativity as well. In a work just accepted to IJCAR 2022, we have closed the cycle by considering associativity. In this proposal, we will show two undecidability results for fragments/variations of acLL_Σ in [5], and present a preliminary focused version for that system.

Introduction. Resource aware logics have been object of passionate study for quite some time now. The motivations for this passion vary: resource consciousness are adequate for modeling steps of computation; logics have interesting algebraic semantics; calculi have nice proof theoretic properties; multi-modalities allow for the specification of several behaviors; there are many interesting applications in linguistics, etc.

With this variety of subjects, applications and views, it is not surprising that different groups developed different systems based on different principles. For example, the Lambek calculus (L) [10] was introduced for mathematical modeling of natural language syntax, and it extends a basic categorial grammar [2, 4] by a concatenation operator. Linear logic (LL) [9], originally discovered by Girard from a semantical analysis of the models of polymorphic λ-calculus, turned out to be a refinement of classical and intuitionistic logic, having the dualities of the former and constructive properties of the latter. The key point is the presence of the *modalities* !, ?, called *exponentials* in LL. In the intuitionistic version of LL, denoted by ILL, only the ! exponential is present.

L and LL were compared in [1], when Abrusci showed that Lambek calculus coincides with a variant of the non-commutative, multiplicative version of ILL [11]. This correspondence can be lifted for considering also the additive connectives: Full (multiplicative-additive) Lambek calculus FL relates to non-commutative multiplicative-additive version of ILL, here denoted by cLL.

In the paper just accepted to IJCAR [5], we have proposed the sequent based system acLL_Σ, a conservative extension of cLL, where associativity is allowed only for formulas marked with a special kind of modality, determined by a *subexponential signature* Σ. The core fragment of acLL_Σ (*i.e.*, without the subexponentials) corresponds to the non-associative version of full Lambek calculus, FNL [6]. This extended abstract presents the two undecidability results of [5] and proposes a focused version for acLL_Σ.

*Speaker.

Non-associative, non-commutative multi-modal linear logic. The language of acLL_Σ consists of a denumerable infinite set of propositional variables $\{p, q, r, \ldots\}$, the unities $\{1, \top\}$, the binary connectives for additive conjunction and disjunction $\{\&, \oplus\}$, the non-commutative multiplicative conjunction $\otimes$, the non-commutative linear implications $\{\to, \leftarrow\}$, and the unary subexponentials $!^i$, with i belonging to a pre-ordered set of labels $(I, \preceq)$.

Roughly speaking, subexponentials [8] are substructural multi-modalities. In LL, $!\,A$ indicates that the linear formula A behaves *classically*, that is, it can be contracted *and* weakened. Labeling ! with indices allows moving one step further: The set I can be partitioned so that, in $!^iA$, A can be contracted *and/or* weakened. In this work, we consider not only weakening and contraction, but also commutativity and associativity, all such substructural properties determined by the axioms:

$$\mathsf{C}: \ !^iF \to !^iF \otimes !^iF \quad \mathsf{W}: \ !^iF \to 1 \quad \mathsf{E}: \ (!^iF) \otimes G \equiv G \otimes (!^iF)$$
$$\mathsf{A1}: \ !^iF \otimes (G \otimes H) \to (!^iF \otimes G) \otimes H \quad \mathsf{A2}: \ (G \otimes H) \otimes !^iF \to G \otimes (H \otimes !^iF)$$

The signature Σ of acLL_Σ contains $(I, \preceq)$ together with a function stating which of those axioms are assumed for each label. Pre-ordering the labels (together with an upward closeness requirement) guarantees cut-elimination [5]. Sequents have a *nested structure*, corresponding to trees of formulas, here called *structures*. And rules are applied deeply in such structures. Formally:

Definition 1 (Structured sequents)**.** Structures *are formulas or pairs containing structures:* $\Gamma, \Delta := F \mid (\Gamma, \Gamma)$*, where the constructors may be empty but never a singleton. The notation* $!^j\Gamma$ *will represent a structure where every formula* $F \in \Gamma$ *is such that* $F = !^jF'$*.*

*An n-*ary context $\Gamma\{^1\}\cdots\{^n\}$ *is a context that contains n pairwise distinct numbered* holes $\{\ \}$ *wherever a formula may otherwise occur. Given n contexts* $\Gamma_1, \ldots, \Gamma_n$*, we write* $\Gamma\{\Gamma_1\}\cdots\{\Gamma_n\}$ *for the context where the k-th hole in* $\Gamma\{^1\}\cdots\{^n\}$ *has been replaced by* Γ_k *(for* $1 \le k \le n$*). If* $\Gamma_k = \emptyset$ *the hole is removed. A* structured sequent *(or simply* sequent*) has the form* $\Gamma \Rightarrow F$ *where* Γ *is a structure and* F *is a formula.*

Definition 2 (SDML)**.** *Let* $\mathcal{A}$ *be a set of axioms. A (non-associative/commutative)* simply dependent multimodal logical system *(*SDML*) is given by a triple* $\Sigma = (I, \preccurlyeq, f)$*, where I is a set of indices,* $(I, \preccurlyeq)$ *is a pre-order, and f is a mapping from I to* $2^{\mathcal{A}}$*.*
If Σ *is a* SDML*, then the* logic described by Σ *has the modality* $!^i$ *for every* $i \in I$*, with the rules of* FNL *depicted in Fig. 1, together with rules for the axioms* $f(i)$ *and the interaction axioms* $!^jA \to !^iA$ *for every* $i, j \in I$ *with* $i \preccurlyeq j$*. Finally, every* SDML *is assumed to be upwardly closed w.r.t.* $\preceq$*, that is, if* $i \preceq j$ *then* $f(i) \subseteq f(j)$ *for all* $i, j \in I$*.*

Fig. 2 presents the structured system acLL_Σ, for the logic described by the SDML determined by Σ, with $\mathcal{A} = \{\mathsf{C}, \mathsf{W}, \mathsf{A1}, \mathsf{A2}, \mathsf{E}\}$ where, in the subexponential rule for $\mathsf{S} \in \mathcal{A}$, the respective $s \in I$ is such that $\mathsf{S} \in f(s)$ (*e.g.* the subexponential symbol e indicates that $\mathsf{E} \in f(e)$). As usual, $\Gamma^{\preceq i}$ represents the context with the tree structure inherited by Γ, with all the subexponentials greater or equal to i.

(Un)decidability results. Non-associativity makes a significant difference in decidability and complexity matters. For our system acLL_Σ, its decidability or undecidability depends on its signature Σ. If for every $i \in I$, $\mathsf{C} \notin f(s)$, then acLL_Σ is clearly decidable, since the cut-free proof search space is finite. Therefore, for undecidability it is necessary to have at least one subexponential which allows contraction.

For FNL with only one fully-powered exponential modality s, undecidability was proven in a preprint by Tanaka [12]. In [5], we have refined Tanaka's result by showing that acLL_Σ containing the multiplicatives $\otimes, \to$, the additive $\oplus$ and one *classical* subexponential is undecidable.

Theorem 1. *If there exists such* $s \in I$ *that* $f(s) \supseteq \{\mathsf{C}, \mathsf{W}\}$*, then the derivability problem in* acLL_Σ *is undecidable. Moreover, this holds for the fragment with only* $\otimes$*,* $\to$*,* $\oplus$*,* $!^s$*.*

In the second undecidability result, we keep two subexponentials, but with a minimalist configuration: the implicational fragment of the logic plus two subexponentials: the "main" one allowing for contraction,

$$\frac{\Gamma\{(F,G)\} \Rightarrow H}{\Gamma\{F \otimes G\} \Rightarrow H}\ \otimes L \qquad \frac{\Gamma_1 \Rightarrow F \quad \Gamma_2 \Rightarrow G}{(\Gamma_1,\Gamma_2) \Rightarrow F \otimes G}\ \otimes R \qquad \frac{\Gamma\{F\} \Rightarrow H \quad \Gamma\{G\} \Rightarrow H}{\Gamma\{F \oplus G\} \Rightarrow H}\ \oplus L$$

$$\frac{\Gamma \Rightarrow F_i}{\Gamma \Rightarrow F_1 \oplus F_2}\ \oplus R_i \qquad \frac{\Gamma\{F_i\} \Rightarrow G}{\Gamma\{F_1 \,\&\, F_2\} \Rightarrow G}\ \& L_i \qquad \frac{\Gamma \Rightarrow F \quad \Gamma \Rightarrow G}{\Gamma \Rightarrow F \,\&\, G}\ \& R$$

$$\frac{\Delta \Rightarrow F \quad \Gamma\{G\} \Rightarrow H}{\Gamma\{(\Delta, F \to G)\} \Rightarrow H}\ \to L \qquad \frac{(F,\Gamma) \Rightarrow G}{\Gamma \Rightarrow F \to G}\ \to R \qquad \frac{\Delta \Rightarrow F \quad \Gamma\{G\} \Rightarrow H}{\Gamma\{(G \leftarrow F, \Delta)\} \Rightarrow H}\ \leftarrow L$$

$$\frac{(\Gamma, F) \Rightarrow G}{\Gamma \Rightarrow G \leftarrow F}\ \leftarrow R \qquad \frac{\Gamma\{\,\} \Rightarrow F}{\Gamma\{1\} \Rightarrow F}\ 1L \qquad \frac{}{\Rightarrow 1}\ 1R \qquad \frac{}{\Gamma \Rightarrow \top}\ \top R$$

$$\frac{}{F \Rightarrow F}\ \mathsf{init} \qquad \frac{\Delta \Rightarrow F \quad \Gamma\left\{{}^{1}F\right\}\ldots\left\{{}^{n}F\right\} \Rightarrow G}{\Gamma\left\{{}^{1}\Delta\right\}\ldots\left\{{}^{n}\Delta\right\} \Rightarrow G}\ \mathsf{mcut}$$

Figure 1: Structured system FNL for non-associative, full Lambek calculus.

$$\frac{\Gamma^{\leq i} \Rightarrow F}{\Gamma \Rightarrow\, !^{i}F}\ !^{i}R \qquad \frac{\Gamma\{F\} \Rightarrow G}{\Gamma\left\{!^{i}F\right\} \Rightarrow G}\ \mathsf{der}$$

$$\frac{\Gamma\{((!^{a}\Delta_1,\Delta_2),\Delta_3)\} \Rightarrow G}{\Gamma\{(!^{a}\Delta_1,(\Delta_2,\Delta_3))\} \Rightarrow G}\ \mathsf{A1} \qquad \frac{\Gamma\{(\Delta_1,(\Delta_2,!^{a}\Delta_3))\} \Rightarrow G}{\Gamma\{((\Delta_1,\Delta_2),!^{a}\Delta_3)\} \Rightarrow G}\ \mathsf{A2} \qquad \frac{\Gamma\{(\Delta_2,!^{e}\Delta_1)\} \Rightarrow G}{\Gamma\{(!^{e}\Delta_1,\Delta_2)\} \Rightarrow G}\ \mathsf{E1}$$

$$\frac{\Gamma\{(!^{e}\Delta_2,\Delta_1)\} \Rightarrow G}{\Gamma\{(\Delta_1,!^{e}\Delta_2)\} \Rightarrow G}\ \mathsf{E2} \qquad \frac{\Gamma\{\,\} \Rightarrow G}{\Gamma\{!^{w}\Delta\} \Rightarrow G}\ \mathsf{W} \qquad \frac{\Gamma\left\{{}^{1}\,!^{c}\Delta\right\}\ldots\left\{{}^{n}\,!^{c}\Delta\right\} \Rightarrow G}{\Gamma\left\{{}^{1}\,\right\}\ldots\left\{{}^{k}\,!^{c}\Delta\right\}\ldots\left\{{}^{n}\,\right\} \Rightarrow G}\ \mathsf{C}$$

Figure 2: Structured system acLL_Σ for the logic described by Σ.

exchange, and associativity (weakening is optional), and an "auxiliary" one allowing only associativity. This is a variation of Chaudhuri's result [7] (in the non-associative, non-commutative case), making use of fewer connectives (tensor is not needed) and less powerful subexponentials.

Theorem 2. *If there are $a, c \in I$ such that $f(a) = \{\mathsf{A1}, \mathsf{A2}\}$ and $f(c) \supseteq \{\mathsf{C}, \mathsf{E}, \mathsf{A1}, \mathsf{A2}\}$, then the derivability problem in acLL_Σ is undecidable. Moreover, this holds for the fragment with only $\to$, $!^{a}$, $!^{c}$.*

Focusing. The focusing discipline [3] is determined by the alternation of *focused* and *unfocused* phases in the proof construction. In the unfocused phase, inference rules can be applied eagerly and no backtracking is necessary; in the focused phase, on the other hand, either context restrictions apply, or choices within inference rules can lead to failures for which one may need to backtrack. These phases are totally determined by the polarities of formulas: provability is preserved when applying right/left rules for negative/positive formulas respectively, but not necessarily in other cases.

The importance of focusing is due to the fact that it gives a notion of *normal forms* for proofs. In the case of acLL_Σ, the following polarization is proposed.

Definition 3 (Polarized Syntax)**.** *Let $\mathcal{P}$ be the set propositional variables and $\mathcal{P}^+ \cap \mathcal{P}^-$ a partition of $\mathcal{P}$, with $A^+ \in \mathcal{P}^+$ and $A^- \in \mathcal{P}^-$. The* polarized formulas *are given by the following grammar*

$$\begin{array}{lcl@{\qquad}lcl} P, Q & := & A^+ \mid 1 \mid F \otimes F \mid F \oplus F \mid F \to F \mid F \leftarrow F \mid\, !^{i}F & N, M & := & A^- \mid \top \mid F \,\&\, F \\ L & := & A^+ \mid N & R & := & A^- \mid P \end{array}$$

A negative structure*, denoted by* Λ*, is given by* $\Lambda := L \mid (\Lambda, \Lambda)$*. A* polarized structured sequent *has one of the forms:* $\Gamma \Rightarrow F \quad \Lambda\{\langle F\rangle\} \Rightarrow R \quad \Lambda \Rightarrow \langle F\rangle$ *where the first is an* unfocused sequent *and the last two are* focused*, with* $\langle F\rangle$ *indicating that the formula* F *is under focus.*

The proposed focused system facLL_Σ is depicted in Figure 3, where the structural rules are restricted to neutral formulas only. Our ongoing work is to show that facLL_Σ is sound and complete w.r.t. acLL_Σ. We plan to apply the result in the analysis of natural language syntax.

$$\frac{\Lambda_1 \Rightarrow \langle F\rangle \quad \Lambda_2 \Rightarrow \langle G\rangle}{(\Lambda_1, \Lambda_2) \Rightarrow \langle F \otimes G\rangle}\ \otimes R \qquad \frac{\Lambda \Rightarrow \langle F_i\rangle}{\Lambda \Rightarrow \langle F_1 \oplus F_2\rangle}\ \oplus R_i \qquad \frac{\Lambda\{\langle F_i\rangle\} \Rightarrow R}{\Lambda\{\langle F_1 \,\&\, F_2\rangle\} \Rightarrow R}\ \& L_i$$

$$\frac{\Lambda' \Rightarrow \langle F\rangle \quad \Lambda\{\langle G\rangle\} \Rightarrow R}{\Lambda\{\langle(\Lambda', F \to G)\rangle\} \Rightarrow R}\ \to L \qquad \frac{\Lambda' \Rightarrow \langle F\rangle \quad \Lambda\{\langle G\rangle\} \Rightarrow R}{\Lambda\{\langle(G \leftarrow F, \Lambda')\rangle\} \Rightarrow R}\ \leftarrow L$$

$$\frac{\Lambda^{\leq i} \Rightarrow F}{\Lambda \Rightarrow \left\langle !^i F\right\rangle}\ !^i R \qquad \frac{}{\Rightarrow \langle 1\rangle}\ 1R \qquad \frac{}{P \Rightarrow \langle P\rangle}\ \mathsf{init}+ \qquad \frac{}{\langle N\rangle \Rightarrow N}\ \mathsf{init}-$$

$$\frac{\Gamma\{(F,G)\} \Rightarrow H}{\Gamma\{F \otimes G\} \Rightarrow H}\ \otimes L \qquad \frac{\Gamma\{F\} \Rightarrow H \quad \Gamma\{G\} \Rightarrow H}{\Gamma\{F \oplus G\} \Rightarrow H}\ \oplus L \qquad \frac{\Gamma \Rightarrow F \quad \Gamma \Rightarrow G}{\Gamma \Rightarrow F \,\&\, G}\ \& R$$

$$\frac{(F,\Gamma) \Rightarrow G}{\Gamma \Rightarrow F \to G}\ \to R \qquad \frac{(\Gamma, F) \Rightarrow G}{\Gamma \Rightarrow G \leftarrow F}\ \leftarrow R \qquad \frac{\Gamma\{\,\} \Rightarrow F}{\Gamma\{1\} \Rightarrow F}\ 1L \qquad \frac{}{\Gamma \Rightarrow \top}\ \top R$$

$$\frac{\Lambda\{\langle N\rangle\} \Rightarrow R}{\Lambda\{N\} \Rightarrow R}\ \mathsf{D}L \qquad \frac{\Lambda \Rightarrow \langle P\rangle}{\Lambda \Rightarrow P}\ \mathsf{D}R \qquad \frac{\Lambda\{\langle F\rangle\} \Rightarrow R}{\Lambda\left\{!^i F\right\} \Rightarrow R}\ \mathsf{der} \qquad \frac{\Lambda\{P\} \Rightarrow R}{\Lambda\{\langle P\rangle\} \Rightarrow R}\ \mathsf{R}L \qquad \frac{\Lambda \Rightarrow N}{\Lambda \Rightarrow \langle N\rangle}\ \mathsf{R}R$$

Figure 3: Structured system facLL_Σ for focused acLL_Σ.

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