Kites and pseudo MV-algebras

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Abstract

We investigate the structure of perfect residuated lattices, focussing especially on perfect pseudo MV-algebras. We show that perfect pseudo MV-algebras can be represented as a generalised version of kites from [8]. We characterise varieties generated by kites and describe the lattice of these varieties as a complete sublattice of the lattice of perfectly generated varieties of perfect pseudo MV-algebras.

1 Introduction

We work in the framework of *residuated lattices*, that is, algebras $\mathbf{A} = (A; \wedge, \vee, \cdot, \backslash, /, 1)$ such that $(A; \wedge, \vee)$ is a lattice, $(A; \cdot, 1)$ is a monoid, and the equivalences

$$y \leq x \backslash z \quad \Leftrightarrow \quad xy \leq z \quad \Leftrightarrow \quad x \leq z / y$$

hold for all $x, y, z \in A$, where the ordering relation $\leq$ is the natural lattice order on A, and multiplication is written as juxtaposition. A residuated lattice expanded by an additional constant 0 is an *FL-algebra* (for **F**ull **L**ambek calculus), and an FL-algebra satisfying $0 \leq x \leq 1$ is an FL_w-algebra.

Our general terminology and notation is that of universal algebra, with a minimum of category theory. For the theory of residuated lattices and all concepts not defined below, we refer the reader to [9], from where we also adopt the convention of using calligraphic letters as variables for arbitrary classes of algebras, and sans-serif for the acronyms of named classes. The acronyms themselves also come from [9], with the exception of the variety of pseudo MV-algebras which we call $\mathsf{\Psi MV}$, and not psMV as in [9].

The present work grew out of an attempt at answering Question 8.4 from [8], concerning a construction of certain algebras called *kites*, most naturally associated with a noncommutative generalisation of BL-algebras known as *pseudo BL-algebras* (see also [6]). The construction has also been used in a broader context of residuated lattices (e.g., [2]) and algebras related to quantum computation (e.g., [7], [1] and [5]). Here we narrow the focus to *pseudo MV-algebras*, and for the most part indeed to *perfect pseudo MV-algebras*. This narrowing of view bears fruit: we obtain several structural results that we believe would be much more difficult to discover (or do not hold at all) in a broader context. We begin however in broad strokes, by establishing a few facts about *perfect residuated lattices*.

Definition 1. *An FL_w-algebra $\mathbf{A}$ is* perfect *if there is a homomorphism $h_{\mathbf{A}} \colon \mathbf{A} \to \mathbf{2}$ such that for any $x \in h_{\mathbf{A}}^{-1}(0)$ and any $y \in h_{\mathbf{A}}^{-1}(1)$ the inequality $x \leq y$ holds.*

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[*] Speaker.

We say that a variety $\mathcal{V}$ of FL_w-algebras is *perfectly generated* if it is generated by its perfect members. Let $\mathbf{A}$ be an FL-algebra, and $a, b \in A$. The *left conjugate* of $a \in A$ by $b \in A$ is the element $\lambda_b(a) := (b \setminus ab) \wedge 1$ and the *right conjugate* is $\rho_b(a) := (ba \,/\, b) \wedge 1$. A *conjugation polynomial* $\boldsymbol{\alpha}$ over $\mathbf{A}$ is any unary polynomial $(\gamma_{a_1} \circ \gamma_{a_2} \circ \cdots \circ \gamma_{a_n})(x)$ where $\gamma \in \{\lambda, \rho\}$ and $a_i \in A$ for $1 \leq i \leq n$. We write $\mathrm{cPol}(\mathbf{A})$ for the set of all conjugation polynomials over $\mathbf{A}$. For an element $u \in A$, an *iterated conjugate* of u is $\boldsymbol{\alpha}(u)$ for some $\boldsymbol{\alpha} \in \mathrm{cPol}(\mathbf{A})$.

Theorem 1. *A subvariety $\mathcal{V}$ of $\mathsf{FL_w}$ is perfectly generated if and only if $\mathcal{V}$ is nontrivial and satisfies the following identities:*

$$\boldsymbol{\alpha}(x \,/\, x^-) \vee \boldsymbol{\beta}(x^- \,/\, x) = 1, \tag{1}$$

$$\boldsymbol{\alpha}((x \vee x^-) \cdot (y \vee y^-))^- \leq \boldsymbol{\alpha}((x \vee x^-) \cdot (y \vee y^-)), \tag{2}$$

$$x \wedge x^- \leq y \vee y^- \tag{3}$$

for every $\mathbf{A} \in \mathcal{V}$ and all $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathrm{cPol}(\mathbf{A})$.

2 Kites and perfect pseudo MV-algebras

As we already mentioned, $\mathsf{\Psi MV}$ will stand for the variety of *pseudo MV-algebras*. We will write $\mathsf{pf\Psi MV}$ for the class of perfect members of $\mathsf{\Psi MV}$, and $\mathsf{P\Psi MV}$ for the variety generated by $\mathsf{pf\Psi MV}$. Now we define a generalised version of a kite.

Definition 2. *Let $\mathbf{L}$ be an ℓ-group and $\lambda\colon \mathbf{L} \to \mathbf{L}$ be an automorphism. We define the algebra*

$$\mathcal{K}(\mathbf{L}, \lambda) := (L^- \uplus L^+; \wedge, \vee, \odot, \setminus, /, 0, 1)$$

where $L^- \uplus L^+$ is a disjoint union, $0 := e \in L^+$, $1 := e \in L^-$, and the other operations are given by

$$x \wedge y := \begin{cases} x \wedge y & \text{if } x, y \in L^-, \\ x & \text{if } x \in L^+, y \in L^- \\ y & \text{if } x \in L^-, y \in L^+, \\ x \wedge y & \text{if } x, y \in L^+, \end{cases} \qquad x \vee y := \begin{cases} x \vee y & \text{if } x, y \in L^-, \\ y & \text{if } x \in L^+, y \in L^- \\ x & \text{if } x \in L^-, y \in L^+, \\ x \vee y & \text{if } x, y \in L^+, \end{cases}$$

$$x \odot y := \begin{cases} x \cdot y & \text{if } x, y \in L^-, \\ \lambda(x) \cdot y \vee e & \text{if } x \in L^-, y \in L^+ \\ x \cdot y \vee e & \text{if } x \in L^+, y \in L^-, \\ e & \text{if } x, y \in L^+, \end{cases}$$

$$x \setminus y := \begin{cases} x^{-1} \cdot y \wedge e & \text{if } x, y \in L^-, \\ e & \text{if } x \in L^+, y \in L^- \\ \lambda(x)^{-1} \cdot y \vee e & \text{if } x \in L^-, y \in L^+, \\ x^{-1} \cdot y \wedge e & \text{if } x, y \in L^+, \end{cases} \qquad y \,/\, x := \begin{cases} y \cdot x^{-1} \wedge e & \text{if } x, y \in L^-, \\ e & \text{if } x \in L^+, y \in L^- \\ y \cdot x^{-1} \vee e & \text{if } x \in L^-, y \in L^+, \\ \lambda^{-1}(y \cdot x^{-1}) \wedge e & \text{if } x, y \in L^+, \end{cases}$$

Remark 1. *The negations $x^- := 0 \,/\, x$ and $x^\sim := x \setminus 0$ in $\mathcal{K}(\mathbf{L}, \lambda)$ are given by*

$$x^- = \begin{cases} x^{-1} & \text{if } x \in L^-, \\ \lambda^{-1}(x)^{-1} & \text{if } x \in L^+. \end{cases} \qquad x^\sim = \begin{cases} \lambda(x)^{-1} \in L^+ & \text{if } x \in L^-, \\ x^{-1} \in L^- & \text{if } x \in L^+. \end{cases}$$

In any perfect pseudo MV-algebra $\mathbf{A}$ the normal filter $F_{\mathbf{A}}$ is the universe of a cancellative IGMV-algebra $\mathbf{F_A}$. It is well known that $\mathbf{F_A}$ uniquely determines an ℓ-group $\ell(\mathbf{F_A})$; indeed ℓ is a functor from the category $\mathsf{CanIGMV}$ of cancellative IGMV-algebras to the category LG of ℓ-groups. Since pseudo MV-algebras satisfy the identities

$$(x \star y)^{\sim\sim} = x^{\sim\sim} \star y^{\sim\sim} \qquad x^{-\sim\sim} = x^{\sim\sim-}$$

where $\star \in \{\wedge, \vee, \cdot\}$, the map $-^{\sim\sim}$ is an automorphism of $\mathbf{F_A}$. Applying the functor ℓ we lift $-^{\sim\sim}$ to an automorphism

$$\ell^{\approx}\colon \ell(\mathbf{F_A}) \to \ell(\mathbf{F_A})$$

defined as $\ell^{\approx}(-) := \ell(-^{\sim\sim})$.

Theorem 2. *Let $\mathbf{A}$ be a perfect pseudo MV-algebra. Then $\mathbf{A} \cong \mathcal{K}(\ell(\mathbf{F_A}), \ell^{\approx})$.*

It was shown in [4] that perfect MV-algebras are categorically equivalent to Abelian ℓ-groups, and in [3] the result was generalised to a categorical equivalence between *symmetric* perfect pseudo MV-algebras and ℓ-groups. We generalise both results below.

Theorem 3. *The categories of perfect pseudo MV-algebras, and of ℓ-groups with a distinguished automorphism, are equivalent. If the distingushed automorphism is the identity, the equivalent category is that of symmetric perfect pseudo MV-algebras.*

3 Varieties generated by kites

For any ℓ-group $\mathbf{L}$, and any bijection $\beta\colon B \to B$, a very natural automorphism $\lambda\colon \mathbf{L}^B \to \mathbf{L}^B$ is induced by taking $\lambda(x(i)) := x(\beta(i))$ for each $i \in B$. Then $\mathcal{K}(\mathbf{L}^B, \lambda)$ is a perfect pseudo MV-algebra.

Definition 3. *A monounary algebra $\mathbf{B} = (B; \beta)$ where β is a bijection on B will be called a* B-cycle*. Homomorphisms of B-cycles are maps $f\colon \mathbf{B} \to \mathbf{C}$ satisfying $f \circ \lambda^{\mathbf{B}} = \lambda^{\mathbf{C}} \circ f$. Objects of the category BC are B-cycles and arrows are homomorphisms.*

The definition below is equivalent to the original definition of a kite from [8].

Definition 4. *Let $\mathbf{B} = (B; \beta)$ be a B-cycle and $\mathbf{L}$ and ℓ-group. A* kite *over $\mathbf{B}$ and $\mathbf{L}$ is the algebra*

$$\mathcal{K}_{\mathbf{B}}(\mathbf{L}) := \mathcal{K}(\mathbf{L}, \lambda)$$

where $\lambda\colon \mathbf{L}^B \to \mathbf{L}^B$ is the automorphism given by $\lambda(x(i)) = x(\beta(i))$ for any $i \in B$.

We write $\Lambda(\mathcal{V})$ for the lattice of subvarieties of a variety $\mathcal{V}$, and $\Lambda^+(\mathcal{V})$ for the poset of nontrivial subvarieties of $\mathcal{V}$. Since the variety BA of Boolean algebras is the unique atom of $\Lambda(\mathsf{P\Psi MV})$, we have that $\Lambda^+(\mathsf{P\Psi MV})$ is a (complete algebraic) sublattice of $\Lambda(\mathsf{P\Psi MV})$.

For any pseudo MV-algebra $\mathbf{A}$, the operation $-^{\approx}$ is a bijection on A, so for any $\mathbf{A}$ we define the dimension of $\mathbf{A}$ to be $dim(-^{\approx})$. From now on, $\mathbb{D}$ will stand for the lattice $(\mathbb{N}; |)$ of natural numbers under the divisibility ordering.

Definition 5. *Let $\mathbf{A} \in \mathsf{PMV}$ and $\mathcal{V} \in \mathsf{\Psi MV}$. Then*

1. *$dim(\mathbf{A}) := dim(-^{\approx})$,*
2. *$dim(\mathcal{V}) := \min^{\mathbb{D}}\{dim(\mathbf{A}) \mid n:$ for all $\mathbf{A} \in \mathcal{V}\}$,*

3. $\mathsf{P\Psi MV}_n := \mathsf{P\Psi MV} \cap \mathrm{Mod}\{\lambda^n(x) = x\}$, *for any* $n \in \mathbb{D}$.

It is immediate that $\mathsf{P\Psi MV}_n$ defined in (3) is the largest subvariety of $\mathsf{P\Psi MV}$ of dimension n. Moreover, for all $n, m \in \mathbb{N}$ we have

$$\mathsf{P\Psi MV}_n \subseteq \mathsf{P\Psi MV}_m \text{ if and only if } n \mid m$$

so in particular $\mathsf{P\Psi MV}_0 = \mathsf{P\Psi MV}$.

Definition 6. *We define two pairs of maps*

$$\psi\colon \Lambda(\mathsf{P\Psi MV}) \to \Lambda(\mathsf{CanIGMV}),\ \textit{where } \psi(\mathcal{V}) = V\{\mathbf{F_A} : \mathbf{A} \in \mathcal{V}_{pf}\},$$
$$\Psi\colon \Lambda(\mathsf{P\Psi MV}) \to \Lambda(\mathsf{CanIGMV}) \times \mathbb{D},\ \textit{where } \Psi(\mathcal{V}) = (\psi(\mathcal{V}),\ \mathit{dim}(\mathcal{V})),$$

for any $\mathcal{V} \in \Lambda(\mathsf{P\Psi MV})$ *and*

$$\delta\colon \Lambda(\mathsf{CanIGMV}) \to \Lambda(\mathsf{P\Psi MV}),\ \textit{where } \delta(\mathcal{V}) = V\{\mathbf{A} \in \mathsf{pf\Psi MV} : \mathbf{F_A} \in \mathcal{V}\},$$
$$\Delta\colon \Lambda(\mathsf{CanIGMV}) \times \mathbb{D} \to \Lambda(\mathsf{P\Psi MV}),\ \textit{where } \Delta(\mathcal{V}, n) = \delta(\mathcal{V}) \cap \mathsf{P\Psi MV}_n,$$

for any $\mathcal{V} \in \Lambda(\mathsf{CanIGMV})$ *and* $n \in \mathbb{D}$.

Theorem 4. *Let* $\mathcal{V} \in \Lambda(\mathsf{P\Psi MV})$. *The following are equivalent.*

1. $\mathcal{V}$ *is generated by kites.*
2. $\mathcal{V} = \Delta\Psi(\mathcal{V})$.
3. $\mathcal{V} = \Delta(\mathcal{W}, n)$ *for some* $\mathcal{W} \in \Lambda(\mathsf{CanIGMV})$ *and some* $n \in \mathbb{D}$.

We end by a more detailed description of the lattice of varieties generated by kites. Incidentally, it answers Questions 8.1 and 8.2 from [8] insofar as they apply in this context.

Theorem 5. *Let* $\mathbb{K}$ *be the lattice of subvarieties of* $\mathsf{P\Psi MV}$ *generated by kites.*

$$\mathbb{K} \cong \mathbf{1} \oplus \left(\Lambda^+(\mathsf{CanIGMV}) \times \mathbb{D}\right) \cong \mathbf{1} \oplus \left(\Lambda^+(\mathsf{LG}) \times \mathbb{D}\right)$$

where $\mathbf{1}$ *is the trivial lattice and* $\oplus$ *is the operation of ordinal sum.*

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