On Proof Equivalence for Modal Logics

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The proof theory of modal logics has seen enormous progress during the last three decades. In the couse of the years, several proof systems have been defined for modal logics: nested sequents [8, 23, 19], hyper sequents [7, 17] and labeled systems [22, 21]. Moreover, we understand the relation between display calculus and nested sequents [12] and hyper sequents [13], and we have focused proof systems for classical and intuitionistic modal logics [20, 9, 10]. However, none of these formalisms provide an answer the question

When are two proofs the same?

The standard approach to this question is to define a proof as an equivalence class of sequent calculus derivations modulo these permutations. In this regard, focused proof systems considerably reduce the rule permutations in the sequent calculi by grouping rules into phases and could be considered a good candidate for a notion of proof equivalence. Nevertheless, because of the sequential nature of the focused systems, derivations differing for the order of phases are still considered different proofs even if they could be transformed the one into the other via rule permutations, making this approach unsatisfactory to answer this question.

In this talk we propose an answer to this question by defining a notion of proof equivalence for some modal logics based on the syntax of *combinatorial proofs* presented in [4, 2, 5]. For these logics we define the following notion of proof identity:

Two proofs are the same iff they have the same combinatorial proof.

Combinatorial proof are a proof system introduced by Hughes in [15, 16], providing a proof system to address the question of proof identity for classical propositional logic and Hilbert's 24th problem [26, 25]. They capture rules permutations, among which as the ones required in the cut-elimination procedure of sequent calculus, and allow to compare proofs in different proof formalisms (see Figure 1), such as sequent calculus [16, 15], calculus of structures [24], resolution calculus and analytic tableaux [3].

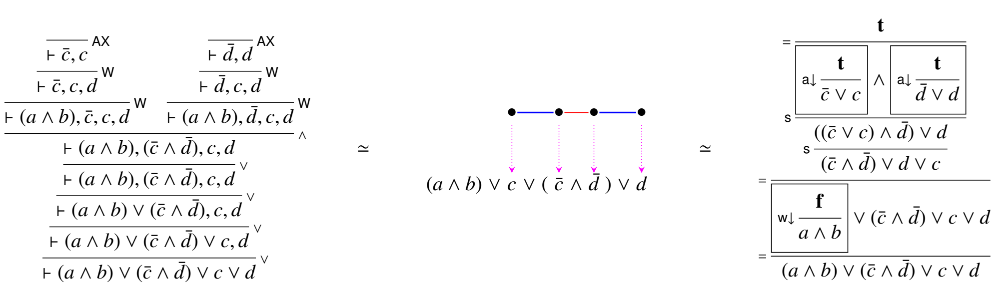

Figure 1: A sequent calculus proof, a combinatorial proof, and a deep inference derivation representing the same proof.

A combinatorial proof is defined as a specific graph homomorphism (the dotted lines in Figure 2) from a graph provided with a partition on its vertices (the blue edges in Figure 2) satisfying certain topological conditions, to a graph encoding a formula (represented by the formula itself in Figure 2).

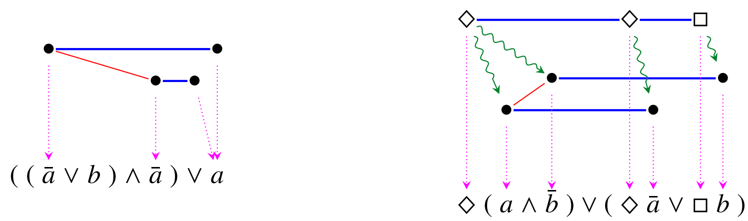

Figure 2: A combinatorial proof of Pierce's law $((\bar{a} \vee b) \wedge \bar{a}) \vee a$ and a combinatorial proof the formula $\Diamond(a \wedge \bar{b}) \vee \Diamond\bar{a} \vee \Box b$, that is, the proof of the axiom $\mathbf{K} \coloneqq \Box(a \supset b) \supset (\Box a \supset \Box b)$.

More precisely, the graph homomorphism captures the resource management part of a proof, that is, resources erasing and duplications. In case of modalities satisfying the the modal axioms **T** and **4**, the homomorphism also capture these transformations. The partitioned graphs represents a linear proof, that is, a proof where each atom is used exactly once. For modal logic, the partition also gather the modalities corresponding to the same rule for the modal axioms **K** and **D**. The partition satisfies specific topological conditions, guaranteeing the possibility of reconstruct a correct derivation using the information contained by the graph.

In this talk we present combinatorial proofs for the modal logics S4-plane and, more in general, non-normal monotonic logics (see the S4-tesseract in [18]). To show soundness and completeness results, we prove a decomposition theorem for these logics by defining hybrid sequent calculi making use of certain deep inference rules [6]. We then prove that the linear part and the resource management part of the proof constructed using this theorem are in correspondence with the partitioned graph and the graph homomorphism of a combinatorial proof. To conclude, we prove that the topological conditions characterising combinatorial proofs can be checked in polynomial time with respect to the size of the graphs, that is, combinatorial proofs are a proof system in the sense of Cook Reckhow [11].

We then present the syntax of combinatorial proofs for intuitionistic logic [14], which relies on similar methods, but different formula encodings and topological conditions, and the combinatorial proofs for the constructive modal logic of the S4-plane [2]. For these combinatorial proofs, we prove a full completeness result (not achievable in the classical setting), and we highlight their relations with winning innocent strategies from game semantics [1].

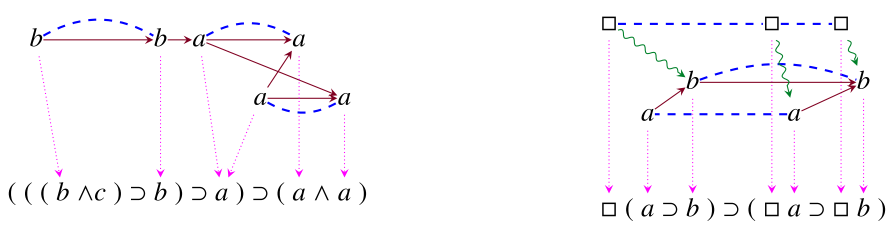

Figure 3: A combinatorial proof of the formula $(((b \wedge c) \supset b) \supset a) \supset (a \wedge a)$ and a combinatorial proof of the modal axiom $\mathbf{K} \coloneqq \Box(a \supset b) \supset (\Box a \supset \Box b)$.

We conclude the presentation with some remarks on the proof equivalence for modal logics induced by this syntax. In particular, we show which rule permutations of the sequent calculus are captured by the proof equivalence defined by the combinatorial proofs syntax, and we explain why certain rule permutations make the complexity of the equivalence problem for these logic non-polynomial. As consequence of this later result, we rule out the possibility of a syntax which, at the same time, is a proof system (in the sense of [11]) and captures all rule permutations.

References

[1] Matteo Acclavio, Davide Catta, and Lutz Straßburger. Game semantics for constructive modal logic. In *International Conference on Automated Reasoning with Analytic Tableaux and Related Methods*, pages 428–445. Springer, 2021.

[2] Matteo Acclavio, Davide Catta, and Lutz Straßburger. Towards a Denotational Semantics for Proofs in Constructive Modal Logic. preprint, April 2021.

[3] Matteo Acclavio and Lutz Straßburger. From syntactic proofs to combinatorial proofs. In *International Joint Conference on Automated Reasoning*, pages 481–497. Springer, 2018.

[4] Matteo Acclavio and Lutz Straßburger. On combinatorial proofs for modal logic. In *International Conference on Automated Reasoning with Analytic Tableaux and Related Methods*, pages 223–240. Springer, 2019.

[5] Matteo Acclavio and Lutz Straßburger. Combinatorial proofs for constructive modal logic, 2022. Accepted at AiML2022.

[6] Andrea Aler Tubella and Lutz Straßburger. Introduction to deep inference. Lecture, August 2019.

[7] Arnon Avron. The method of hypersequents in the proof theory of propositional non-classical logics. In *Logic: from foundations to applications, European Logic Colloquium*, pages 1–32. Oxford University Press, 1994.

[8] Kai Brünnler. Deep sequent systems for modal logic. *Archive for Mathematical Logic*, 48(6):551–577, 2009.

[9] Kaustuv Chaudhuri, Sonia Marin, and Lutz Straßburger. Focused and synthetic nested sequents. In *FoSSaCS'16*, pages 390–407. Springer, 2016.

[10] Kaustuv Chaudhuri, Sonia Marin, and Lutz Straßburger. Modular Focused Proof Systems for Intuitionistic Modal Logics. In Delia Kesner and Brigitte Pientka, editors, *FSCD'16*, volume 52 of *LIPIcs*, pages 16:1–16:18. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, 2016.

[11] Stephen A. Cook and Robert A. Reckhow. The relative efficiency of propositional proof systems. *J. of Symb. Logic*, 44(1):36–50, 1979.

[12] Rajeev Goré, Linda Postniece, and Alwen Tiu. On the correspondence between display postulates and deep inference in nested sequent calculi for tense logics. *Logical Methods in Computer Science*, 7(2), 2011.

[13] Rajeev Goré, Revantha Ramanayake, et al. Labelled tree sequents, tree hypersequents and nested (deep) sequents. *Advances in modal logic*, 9:279–299, 2012.

[14] Willem B Heijltjes, Dominic JD Hughes, and Lutz StraBburger. Intuitionistic proofs without syntax. In *2019 34th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*, pages 1–13. IEEE, 2019.

[15] Dominic Hughes. Proofs Without Syntax. *Annals of Math.*, 164(3):1065–1076, 2006.

[16] Dominic Hughes. Towards Hilbert's 24^{th} problem: Combinatorial proof invariants: (preliminary version). *Electr. Notes Theor. Comput. Sci.*, 165:37–63, 2006.

[17] Björn Lellmann. Hypersequent rules with restricted contexts for propositional modal logics. *Theoretical Computer Science*, 656:76–105, 2016.

[18] Björn Lellmann and Elaine Pimentel. Modularisation of sequent calculi for normal and non-normal modalities. *ACM Transactions on Computational Logic (TOCL)*, 20(2):7, 2019.

[19] Sonia Marin and Lutz Straßburger. Label-free Modular Systems for Classical and Intuitionistic Modal Logics. In *Advances in Modal Logic 10*, 2014.

[20] Dale Miller and Marco Volpe. Focused labeled proof systems for modal logic. In *Logic for Programming, Artificial Intelligence, and Reasoning*, pages 266–280. Springer, 2015.

[21] Sara Negri. Proof analysis in modal logic. *Journal of Philosophical Logic*, 34(5-6):507, 2005.

[22] Alex K Simpson. *The proof theory and semantics of intuitionistic modal logic*. PhD thesis, University of Edinburgh. College of Science and Engineering, 1994.

[23] Lutz Straßburger. Cut elimination in nested sequents for intuitionistic modal logics. In Frank Pfenning, editor, *FoSSaCS'13*, volume 7794 of *LNCS*, pages 209–224. Springer, 2013.

[24] Lutz Straßburger. Combinatorial flows and their normalisation. In Dale Miller, editor, *FSCD'17*, volume 84 of *LIPIcs*, pages 31:1–31:17. Schloss Dagstuhl, 2017.

[25] Lutz Straßburger. The problem of proof identity, and why computer scientists should care about Hilbert's 24th problem. *Philosophical Transactions of the Royal Society A*, 377(2140):20180038, 2019.

[26] Rüdiger Thiele. Hilbert's twenty-fourth problem. *American Math. Monthly*, 110:1–24, 2003.