One-Variable Lattice-Valued Logics

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The *one-variable fragment* of any first-order logic yields an "S5-like" modal logic, obtained by replacing each occurrence of an atom $P(x)$ with a propositional variable p, and $(\forall x)$ and $(\exists x)$ with $\Box$ and $\Diamond$, respectively. The first-order semantics typically induces a relational semantics for this modal logic, but finding an axiomatization for its algebraic semantics is hindered by the fact that an axiomatization of the one-variable fragment cannot be directly extracted from an axiomatization of the full logic. Nevertheless, axiomatizations have been obtained in certain well-known cases. Monadic Boolean algebras [12] and monadic Heyting algebras [3,14] correspond to the one-variable fragments of first-order classical logic and intuitionistic logic, respectively. More generally, varieties of monadic Heyting algebras corresponding to one-variable fragments of first-order intermediate logics have been investigated in [1,2,4–6,15,17,18]. One-variable fragments of some first-order many-valued logics have also been studied in some depth; notably, monadic MV-algebras [7,10,16] and monadic Abelian ℓ-groups [13] correspond to the one-variable fragments of first-order Łukasiewicz logic and Abelian logic, respectively.

In [9], we initiate a general approach to addressing this axiomatization problem. Let $\mathcal{L}$ be an algebraic signature containing binary operations $\wedge$ and $\vee$, and consider the sets $\mathrm{Fm}^1_\forall(\mathcal{L})$ of *(first-order) one-variable $\mathcal{L}$-formulas* (with quantifiers $\forall$ and $\exists$) and $\mathrm{Fm}_\Box(\mathcal{L})$ of *propositional modal formulas* (with modalities $\Box$ and $\Diamond$), denoting by $(-)^*$ the standard translation function from $\mathrm{Fm}^1_\forall(\mathcal{L})$ to $\mathrm{Fm}_\Box(\mathcal{L})$. Members of both $\mathrm{Fm}^1_\forall(\mathcal{L})$ and $\mathrm{Fm}_\Box(\mathcal{L})$ are interpreted using semantics based on algebraic structures for the signature $\mathcal{L}$ with a lattice reduct, called *$\mathcal{L}$-lattices*. For $\mathrm{Fm}^1_\forall(\mathcal{L})$, we define structures over complete $\mathcal{L}$-lattices and interpret the quantifiers $\forall$ and $\exists$ as infima and suprema. For $\mathrm{Fm}_\Box(\mathcal{L})$, we call an algebraic structure $\langle \mathbf{A}, \Box, \Diamond\rangle$ an *m-$\mathcal{L}$-lattice* if $\mathbf{A}$ is an $\mathcal{L}$-lattice and $\Box, \Diamond$ are unary operations satisfying

$(\mathrm{L1}_\Box)$	$\Box x \wedge x \approx \Box x$	$(\mathrm{L1}_\Diamond)$	$\Diamond x \vee x \approx \Diamond x$
$(\mathrm{L2}_\Box)$	$\Box(x \wedge y) \approx \Box x \wedge \Box y$	$(\mathrm{L2}_\Diamond)$	$\Diamond(x \vee y) \approx \Diamond x \vee \Diamond y$
$(\mathrm{L3}_\Box)$	$\Box\Diamond x \approx \Diamond x$	$(\mathrm{L3}_\Diamond)$	$\Diamond\Box x \approx \Box x$,

and for each n-ary operation symbol $\star$ of $\mathcal{L}$,

$$(\star_\Box)\quad \Box(\star(\Box x_1, \dots, \Box x_n)) \approx \star(\Box x_1, \dots, \Box x_n).$$

For any class $\mathcal{K}$ of complete $\mathcal{L}$-lattices, semantical sentential consequence $\models^\forall_\mathcal{K}$ is defined over $\mathrm{Fm}^1_\forall(\mathcal{L})$-equations, i.e., formal expressions of the form $\varphi \approx \psi$ where $\varphi, \psi \in \mathrm{Fm}^1_\forall(\mathcal{L})$. Similarly, for any class $\mathcal{M}$ of m-$\mathcal{L}$-lattices, semantical consequence $\models_\mathcal{M}$ is defined over $\mathrm{Fm}_\Box(\mathcal{L})$-equations.

Observe now that any complete $\mathcal{L}$-lattice $\mathbf{A}$ and set W yields an m-$\mathcal{L}$-lattice $\langle \mathbf{A}^W, \Box, \Diamond\rangle$, that we call *full functional*, where the operations of $\mathbf{A}^W$ are defined pointwise and for each $f \in A^W$ and $u \in W$,

$$\Box f(u) = \bigwedge_{v\in W} f(v) \quad \text{and} \quad \Diamond f(u) = \bigvee_{v \in W} f(v).$$

We also call an m-$\mathcal{L}$-lattice *functional* if it embeds into a full functional m-$\mathcal{L}$-lattice.

Given any class $\mathcal{K}$ of complete $\mathcal{L}$-lattices, let $\mathcal{K}_f$ denote the class of all full functional m-$\mathcal{L}$-lattices $\langle \mathbf{A}^W, \Box, \Diamond \rangle$ with $\mathbf{A} \in \mathcal{K}$. It follows easily that for any set of $\mathrm{Fm}^1_\forall(\mathcal{L})$-equations $T \cup \{\varphi \approx \psi\}$ (lifting the translation * to sets of $\mathrm{Fm}^1_\forall(\mathcal{L})$-equations in the obvious way),

$$T \vDash^\forall_\mathcal{K} \varphi \approx \psi \quad \iff \quad T^* \vDash_{\mathcal{K}_f} \varphi^* \approx \psi^*.$$

The general problem addressed here is to provide an (elegant) axiomatization of the generalized quasivariety of m-$\mathcal{L}$-lattices generated by $\mathcal{K}_f$: that is, the class of *all* m-$\mathcal{L}$-lattices $\mathbf{M}$ satisfying $T^* \vDash_\mathbf{M} \varphi^* \approx \psi^*$ whenever $T \vDash^\forall_\mathcal{K} \varphi \approx \psi$ for a set of $\mathrm{Fm}^1_\forall(\mathcal{L})$-equations $T \cup \{\varphi \approx \psi\}$. In this work, we solve this problem for the case where $\mathcal{K}$ is the class of complete members of a variety that satisfies two natural algebraic properties.

Given any class $\mathcal{K}$ of $\mathcal{L}$-lattices, let $\overline{\mathcal{K}}$ denote the class of complete members of $\mathcal{K}$ and let $m\mathcal{K}$ denote the class of m-$\mathcal{L}$-lattices $\langle \mathbf{A}, \Box, \Diamond \rangle$ with $\mathbf{A} \in \mathcal{K}$. Following closely the proof of the same result for monadic Heyting algebras given in [2], we obtain a general *functional representation theorem* that gives sufficient conditions on $\mathcal{K}$ for all algebras in $m\mathcal{K}$ to be functional. Recall that a class $\mathcal{K}$ of $\mathcal{L}$-lattices

(i) admits *regular completions* if for any $\mathbf{A} \in \mathcal{K}$, there exist a $\mathbf{B} \in \overline{\mathcal{K}}$ and an embedding $f \colon \mathbf{A} \to \mathbf{B}$ that preserves all existing meets and joins of $\mathbf{A}$;

(ii) has the *superamalgamation property* if for any $\mathbf{A}, \mathbf{B}_1, \mathbf{B}_2 \in \mathcal{K}$ and embeddings $f_1 \colon \mathbf{A} \to \mathbf{B}_1$, $f_2 \colon \mathbf{A} \to \mathbf{B}_2$, there exist a $\mathbf{C} \in \mathcal{K}$ and embeddings $g_1 \colon \mathbf{B}_1 \to \mathbf{C}$, $g_2 \colon \mathbf{B}_2 \to \mathbf{C}$ such that $g_1 \circ f_1 = g_2 \circ f_2$ and for any $b_1 \in B_1$, $b_2 \in B_2$ and distinct $i, j \in \{1, 2\}$ such that $g_i(b_i) \leq g_j(b_j)$, there exists an $a \in A$ satisfying $g_i(b_i) \leq g_i \circ f_i(a) = g_j \circ f_j(a) \leq g_j(b_j)$.

Theorem 1. *Let $\mathcal{K}$ be a class of $\mathcal{L}$-lattices that is closed under subalgebras and direct limits, admits regular completions, and has the superamalgamation property. Then every member of $m\mathcal{K}$ is functional.*

Combing this functional representation theorem with our previous observation regarding the relationship between consequence in a class of complete $\mathcal{L}$-lattices and the corresponding class of full functional m-$\mathcal{L}$-lattices, we obtain the following result:

Corollary 1. *Let $\mathcal{V}$ be a variety of $\mathcal{L}$-lattices that admits regular completions and has the superamalgamation property. Then for any set $T \cup \{\varphi \approx \psi\}$ of $\mathrm{Fm}^1_\forall(\mathcal{L})$-equations,*

$$T \vDash^\forall_{\overline{\mathcal{V}}} \varphi \approx \psi \quad \iff \quad T^* \vDash_{m\mathcal{V}} \varphi^* \approx \psi^*.$$

In particular, when $\mathcal{V}$ is the variety of Boolean algebras or Heyting algebras, both of which admit regular completions and have the superamalgamation property, $m\mathcal{V}$ is the variety of monadic Boolean algebras [12] or monadic Heyting algebras [14], respectively, and Corollary 1 yields well-known completeness results for the one-variable fragments of first-order classical logic and intuitionistic logic.

Further examples can be taken from the class of substructural logics (see, e.g., [11]). In particular, letting $\mathcal{L}_s$ be a signature with binary connectives $\vee$, $\wedge$, $\cdot$, and $\to$, and constant symbols f and e, an $\mathrm{FL_e}$-*algebra* is an $\mathcal{L}_s$-lattice $\mathbf{A} = \langle A, \vee, \wedge, \cdot, \to, \mathrm{f}, \mathrm{e} \rangle$ such that $\langle A, \cdot, \mathrm{e} \rangle$ is a commutative monoid and $\to$ is the residuum of $\cdot$, i.e., $a \cdot b \leq c \iff a \leq b \to c$ for all $a, b, c \in A$. Let us denote by $\mathcal{FL}_e$ the variety of $\mathrm{FL_e}$-algebras and by $\mathcal{FL}_{ew}$ and $\mathcal{FL}_{ec}$ the subvarieties of $\mathrm{FL_e}$-algebras satisfying $\mathrm{f} \leq x \leq \mathrm{e}$ and $x \leq x \cdot x$, respectively, noting that $\mathcal{FL}_{ew} \cap \mathcal{FL}_{ec}$ is term-equivalent to the variety of Heyting algebras. Since these varieties are closed under MacNeille completions and have the superamalgamation property (see, e.g., [11]), Theorem 1 and Corollary 1 yield the following result:

Theorem 2. *Let $\mathcal{V} \in \{\mathcal{FL}_e, \mathcal{FL}_{ew}, \mathcal{FL}_{ec}\}$. Then any member of $m\mathcal{V}$ is functional and for any set $T \cup \{\varphi, \psi\}$ of $\mathrm{Fm}^1_\forall(\mathcal{L}_s)$-equations,*

$$T \models^\forall_\mathcal{V} \varphi \approx \psi \quad \Longleftrightarrow \quad T \models_{m\mathcal{V}} \varphi^* \approx \psi^*.$$

Note also that it was proved in [8] that a variety of $\mathrm{FL_e}$-algebras axiomatized relative to $\mathcal{FL}_e$ by "$\mathcal{N}_2$-equations" (i.e., equations of a certain simple syntactic form) is closed under MacNeille completions if and only if it has an analytic sequent calculus of a certain form. It is also known that a variety of $\mathrm{FL_e}$-algebras has the superamalgamation property if and only if it has the Craig interpolation property (see, e.g., [11]); however, a precise characterization of which varieties of $\mathrm{FL_e}$-algebras (even those with an analytic sequent calculus) have these properties is not known.

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