What is the cost of cut?

Timo Lang[1], Carlos Olarte[2], and Elaine Pimentel[1]*

[1] University College London, UK
{t.lang,e.pimentel}@ucl.ac.uk

[2] Université Sorbonne Paris Nord, France
olarte@lipn.fr

Abstract

In [8, 7] we looked at substructural calculi from a game semantic point of view, guided by certain intuitions about resource conscious and, more specifically, cost conscious reasoning. This culminated in labelled extensions of (intuitionistic, affine) linear logic with multimodalities (subexponentials), which allowed for an elegant interpretation of the *dereliction* rule. In this work, we investigate the proof theoretical effect of costs in the cut-elimination process.

Introduction. Various kinds of game semantics have been introduced to characterize computational features of substructural logics, in particular fragments and variants of linear logic (LL) [6]. This line of research can be traced back to the works of Blass [3], Abramsky and Jagadeesan [1] among several others.

Our particular view of game semantics is that it is not just a technical tool for characterizing provability in certain calculi, but rather a playground for illuminating specific semantic intuitions underlying certain proof systems. Specially, we aim at a better understanding of *resource conscious* reasoning, which is often cited as a motivation for substructural logics.

As presented in [8], in a first step, we characterize a version of linear logic (exponential-free affine inuitionistic linear logic **aIMALL**, or, equivalently, Full Lambek Calculus with exchange and weakening FLew) by a game, where the difference between additive and multiplicative connectives is modeled as sequential versus parallel continuation in game states that directly correspond to sequents. More precisely, every branching rule for a multiplicative connective corresponds to a game rule that splits the current run of the game into two independent subgames. Player **P**, who seeks to establish the validity of a given sequent, has to win all the resulting subgames. In contrast, a branching rule for an additive connective is modeled by a choice of player **O** between two possible succeeding game states, corresponding to the premises of the sequent rule in question. Note that this amounts to a deviation from the paradigm "formulas as games", underlying the game semantic tradition initiated by Blass [3]. Our games are, at least structurally, closer to Lorenzen's game for intuitionistic logic [9], where a state roughly corresponds to a situation in which a proponent seeks to defend a particular statement against attacks from an opponent, who, in general, has already granted a bunch of other statements. This kind of semantics for linear logic (but without the sequential/parallel distinction) was first explored in [5].

As long as we only care about the existence of winning strategies, the distinction between sequential and parallel subgames is redundant. However, our model not only highlights the intended semantics, but it also has concrete effects once we introduce *costs* for resources (represented by formulas) into the game. This is done via the unary operator $!^a$, $a \in \mathbb{R}^+$, called *subexponential* in LL (SELL [4, 10]). The intuition is that, from $!^a A$ we can obtain A as often as we want, each time paying the price a. We lift our game to the extended language by enriching game states with a *budget* that is decreased whenever a price is paid. Different strategies for proving the same endsequent can then be compared by the budget

*Speaker.

which they require to be run safely, i.e. without getting into debts. This form of resource consciousness not only enhances the game, but it also translates into a novel sequent system, where cost bounds for proofs are attached as labels to sequents. In [8], we only considered resources in *assumptions*. This is translated to sequents by restricting *negatively* the occurrences of the modalities $!^a$. Thus a promotion rule was not present and the proof-theoretic properties of the proposed systems, such as cut-elimination, can be mimicked by the ones of **aIMALL**. Here we move towards two possible generalizations, allowing modalities also in positive contexts: (i) we propose an admissible cut rule for a restricted form of cut formulas; (ii) we propose an alternative notion of the cut rule itself, with a cost-continuation kind of style.

A game model of branching with costs We will denote by $C^\ell(\mathbb{R}^+)$ the SELL system with labelled sequents of the form $b : \Gamma \longrightarrow \Delta$ where Γ, Δ are multisets of formulas and $b \in \mathbb{R}^+$. Formulas are built from the grammar $A ::= p \mid \mathbf{0} \mid \mathbf{1} \mid A_1 \multimap A_2 \mid A_1 \otimes A_2 \mid A_1 \,\&\, A_2 \mid A_1 \oplus A_2 \mid !^a A$, where p stands for atomic propositions (variables); $\mathbf{0}/\mathbf{1}$ are the false/true units; $\multimap$ denotes linear implication; $\otimes/\&$ are the multiplicative/additive conjunctions; $\oplus$ is the additive disjunction; and $!^a A$ is a subexponential with $a \in \mathbb{R}^+$. The rules for $C^\ell(\mathbb{R}^+)$ are depicted in Fig 1.

The game described by $C^\ell(\mathbb{R}^+)$ is formally defined as follows.

Definition 1 (The game $\mathcal{G}_C(\mathbb{R}^+)$)**.** *$\mathcal{G}_C(\mathbb{R}^+)$ is a game of two players,* **P** *and* **O***. Game states are tuples (H, b), where H is a finite multiset of extended sequents and $b \in \mathbb{R}$ is a "budget". $\mathcal{G}_C$ proceeds in rounds, initiated by* **P***'s selection of an extended sequent S from the current game state. The successor state is determined according to rules that fit one of the two following schemes:*

$$\begin{array}{llcl} (1) & (G \cup \{S\}, b) & \rightsquigarrow & (G \cup \{S'\}, b') \\ (2) & (G \cup \{S\}, b) & \rightsquigarrow & (G \cup \{S^1\} \cup \{S^2\}, b) \end{array}$$

A round proceeds as follows: After **P** *has chosen an extended sequent $S \in H$ among the current game state, she chooses a rule instance r of $C(\mathbb{R}^+)$ such that S is the conclusion of that rule. Depending on r, the round proceeds as follows:*

1. *If r is a unary rule different from $!_L$ with premise S', then the game proceeds in the game state $(G \cup \{S'\}, b)$.*
2. **Budget decrease:** *If $r = !_L$ with premise S' and principal formula $!^a A$, then the game proceeds in the game state $(G \cup \{S'\}, b - a)$.*
3. **Parallelism:** *If r is a binary rule with premises S_1, S_2 pertaining to a* multiplicative *connective, then the game proceeds as $(G \cup \{S_1\} \cup \{S_2\}, b)$.*
4. **O-choice:** *If r is a binary rule with premises S_1, S_2 pertaining to an* additive *connective, then* **O** *chooses $S' \in \{S_1, S_2\}$ and the game proceeds in the game state $(G \cup \{S'\}, b)$.*

A **winning state** *(for* **P***) is a game state (H, b) such that all $S \in H$ are initial sequents of $C(\mathbb{R}^+)$ and $b \geq 0$.*

We write $\models_{\mathcal{G}_C(\mathbb{R}^+)} (H, b)$ if **P** has a w.s. in the $\mathcal{G}_C(\mathbb{R}^+)$-game starting on (H, b). The intuitive reading of $\models_{\mathcal{G}_C(\mathbb{R}^+)} (H, b)$ is: *The budget b suffices to win the game H.*

The following result states the strong adequacy for $\mathcal{G}_C(\mathbb{R}^+)$ w.r.t $C^\ell(\mathbb{R}^+)$.

Theorem 1. $\models_{\mathcal{G}_C(\mathbb{R}^+)} (\{\Gamma \longrightarrow A\}, b)$ *iff* $\vdash_{C^\ell(\mathbb{R}^+)} b : \Gamma \longrightarrow A$*.*

The problem with cut-admissibility. Due to the tight relationship between $C^\ell(\mathbb{R}^+)$ and SELL, it is clear that $C(\mathbb{R}^+)$ inherits the admissibility of the following cut rule, for some $c \in \mathbb{R}$.

$$\frac{a : !\Gamma, \Delta_1 \longrightarrow A \quad b : !\Gamma, \Delta_2, A \longrightarrow C}{c : !\Gamma, \Delta_1, \Delta_2 \longrightarrow C}\ cut$$

labelled sequent system for $C^\ell(\mathbb{R}^+)$

$$\frac{b : \Gamma, A, B \longrightarrow C}{b : \Gamma, A \otimes B \longrightarrow C} \otimes_L \quad \frac{a : !\Gamma, \Delta_1 \longrightarrow A \quad b : !\Gamma, \Delta_2 \longrightarrow B}{a + b : !\Gamma, \Delta_1, \Delta_2 \longrightarrow A \otimes B} \otimes_R$$

$$\frac{a : !\Gamma, \Delta_1 \longrightarrow A \quad b : !\Gamma, \Delta_2, B \longrightarrow C}{a + b : !\Gamma, \Delta_1, \Delta_2, A \multimap B \longrightarrow C} \multimap_L \quad \frac{b : \Gamma, A \longrightarrow B}{b : \Gamma \longrightarrow A \multimap B} \multimap_R$$

$$\frac{b : \Gamma, A_i \longrightarrow B}{b : \Gamma, A_1 \,\&\, A_2 \longrightarrow B} \&_{L_i} \quad \frac{a : \Gamma \longrightarrow A \quad b : \Gamma \longrightarrow B}{\max\{a, b\} : \Gamma \longrightarrow A \,\&\, B} \&_R$$

$$\frac{a : \Gamma, A \longrightarrow C \quad b : \Gamma, B \longrightarrow C}{\max\{a, b\} : \Gamma, A \oplus B \longrightarrow C} \oplus_L \quad \frac{b : \Gamma \longrightarrow A_i}{b : \Gamma \longrightarrow A_1 \oplus A_2} \oplus_{R_i}$$

$$\frac{b : \Gamma, !^a A, A \longrightarrow C}{b + a : \Gamma, !^a A \longrightarrow C} !_L \quad \frac{b : \Gamma^{a\leq}, A \longrightarrow A}{b : \Gamma \longrightarrow !^a A} !_R$$

$$\frac{}{b : \Gamma, p \longrightarrow p} I \quad \frac{}{b : \Gamma \longrightarrow \mathbf{1}} \mathbf{1}_R \quad \frac{}{b : \Gamma, \mathbf{0} \longrightarrow A} \mathbf{0}_L \quad \frac{a : \Gamma \longrightarrow A}{b : \Gamma \longrightarrow A} w_\ell (b \geq a)$$

Figure 1: The labelled sequent system $C^\ell(\mathbb{R}^+)$

The question then is: how to calculate c? The following result shows that it is not possible to define a function for determining the label of the conclusion depending exclusively on the labels of the premises.

Theorem 2. *There is no function $f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ such that the following rule is admissible in $C^\ell(\mathbb{R}^+)$.*

$$\frac{!\Gamma, \Delta_1 \longrightarrow_a A \quad !\Gamma, \Delta_2, A \longrightarrow_b C}{!\Gamma, \Delta_1, \Delta_2 \longrightarrow_{f(a,b)} C} \ cut$$

Proof: Let p, q be different propositional variables, and let $A^{\otimes n}$ denote the n-fold multiplicative conjunction of a formula A. The sequents $!^{1/k} p \longrightarrow_a \, !^{1/k} p^{\otimes(k\cdot a)}$ and $!^{1/k} p^{\otimes(k\cdot a)} \longrightarrow_b p^{\otimes(k\cdot k\cdot a\cdot b)}$ are provable in $C^\ell(\mathbb{R}^+)$ for all natural numbers a, b, k. The smallest label f which makes their cut conclusion $!^{1/k} p \longrightarrow_f p^{\otimes(k\cdot k\cdot a\cdot b)}$ provable in $C^\ell(\mathbb{R}^+)$ is $k \cdot a \cdot b$, which is not a function on the premise labels a, b. □

Some alternative paths. We finish this text by discussing two alternatives for defining a notion of cuts with costs: first by restricting the cut-formulas; second by enhancing the notion of cut rule.

The following cut rule is admissible for a restricted form of the cut formula [7].

Theorem 3. *If A is bang-free and $c \neq 0$, then the following cut rule is admissible in $C^\ell(\mathbb{R}^+)$:*

$$\frac{a : !\Gamma, \Delta_1 \longrightarrow !^c A \quad b : !\Gamma, \Delta_2, !^c A \longrightarrow C}{f(a, b, c) : !\Gamma, \Delta_1, \Delta_2 \longrightarrow C} \ cut_\ell \quad \textit{where } f(a, b, c) = b + \lfloor b/c \rfloor .a$$

Note that, in the particular case where the cut formula itself has no bangs from the beginning, then $f(a, b) = a + b$. On the other hand, the general case where A is not bang-free is an open problem.

Finally, Thm. 2 leaves open the possibility that cut is admissible w.r.t. a function f which takes more information of the premises into account than just their labels. The next definition formalizes this process.

Definition 2. *Let $\mathcal{E} = \{a_b \mid a, b \in \mathbb{R}^+\}$ be such that*

1. $a_b +_{\mathcal{E}} c_d = a + b + c + d$.

2. $a_b \geq_{\mathcal{E}} a_c$ *(i.e., the ordering* $\geq_{\mathcal{E}}$ *ignores the subindices).*

3. $a_b >_{\mathcal{E}} c_d$ *iff* $a > c$.

For any formula $F \in C^{\ell}(\mathbb{R}^+)$*, we define* $[F]_c$ *as the formula that substitutes any modality* $!^{a_b}$ *with* $!^{a_{b+c}}$.

Hence $C^{\ell}(\mathbb{R}^+)$ can be slightly modified so that sequent labels belong to $\mathbb{R}^+$, while modal labels belong to $\mathcal{E}$. Due to the ordering above, the promotion of $!^{a_0}$ has the same effect/constraints that the promotion of $!^{a_b}$. However, the dereliction of the latter requires a greater budget ($a + b$ instead of a). Moreover, the equivalence $!^{a_b}F \equiv !^{a_c}F$ can be proven, each direction requiring a different budget. Finally, note that $\mathcal{E}_0 = \{a_0 \mid a \in \mathbb{R}^+\} \simeq \mathbb{R}^+$, that is, each element $a \in \mathbb{R}^+$ can be seen as the equivalence class of a_0 in $\mathbb{R}^+ \times \mathbb{R}^+$ modulo $\mathbb{R}^+$. We will abuse the notation and continue representing the resulting system by $C^{\ell}(\mathbb{R}^+)$, also unchanging the representation of sequents. The following has a straightforward proof.

Lemma 1. *If* $b : \Gamma, [F]_c \longrightarrow G$ *then* $b : \Gamma, F \longrightarrow G$ *with* $b \geq b'$*. More generally, if* $b : \Gamma, [F]_c \longrightarrow C$ *and* $c \geq c'$ *then* $b : \Gamma, [F]_{c'} \longrightarrow C$ *with* $b \geq b'$.

The next definition restricts the appearance of unbounded modalities only under linear implication.

Definition 3. F *is* $\multimap$*-linear if for all subformulas of the form* $A \multimap B$*,* A *doesn't have occurrences of* $!^a$.

The following result presents the admissibility of an extended form of the cut rule, where the budget information from the left premise is passed to the cut-formula in the right premise. Observe that the label of the conclusion is now a function of the labels of the premises.

Theorem 4 ($\multimap$-linear cut)**.** *The following rule is admissible*

$$\frac{a : !\Gamma, \Delta_1 \longrightarrow F \quad b : !\Gamma, \Delta_2, [F]_a \longrightarrow C}{a + b : !\Gamma, \Delta_1, \Delta_2 \longrightarrow C} \; cut_{LL} \quad F \text{ is a } \multimap\text{-linear formula}$$

Moreover, if $a : \Gamma \longrightarrow C$ *is provable using* cut_{LL}*, then there is a cut-free proof of* $a' : \Gamma \longrightarrow C$ *with* $a \geq a'$.

For future work, we expect that the study of costs of proofs and cut-elimination in labeled calculi may indicate a relationship between labels and bounds of computation as in [2].

References

[1] S. Abramsky and R. Jagadeesan. Games and full completeness for multiplicative linear logic. *J. Symb. Log.*, 59(2):543–574, 1994.

[2] B. Accattoli, S. Graham-Lengrand, and D. Kesner. Tight typings and split bounds, fully developed. *J. Funct. Program.*, 30:e14, 2020.

[3] A. Blass. A game semantics for linear logic. *Ann. Pure Appl. Logic*, 56(1-3):183–220, 1992.

[4] V. Danos, J-B. Joinet, and H. Schellinx. The structure of exponentials: Uncovering the dynamics of linear logic proofs. In *Kurt Gödel Colloquium*, volume 713 of *LNCS*, pages 159–171. Springer, 1993.

[5] C. Fermüller and T. Lang. Interpreting sequent calculi as client-server games. In *TABLEAUX 2017*, volume 10501 of *LNCS*, pages 98–113. Springer, 2017.

[6] J-Y. Girard. Linear logic. *Theor. Comput. Sci.*, 50:1–102, 1987.

[7] T. Lang. *Games, Modalities and Analytic Proofs in Nonclassical Logics*. PhD thesis, TU Wien, 2021.

[8] T. Lang, C. Olarte, E. Pimentel, and C. Fermüller. A game model for proofs with costs. In *TABLEAUX 2019*, volume 11714 of *LNCS*, 241–258. Springer, 2019.

[9] P. Lorenzen. Logik und agon. *Atti Del XII Congresso Internazionale di Filosofia*, 4:187–194, 1960.

[10] V. Nigam and D. Miller. Algorithmic specifications in linear logic with subexponentials. In *PPDP*, 129–140. ACM, 2009.