A focused linear nested system for multi-modalities

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Abstract

Linear logic (LL) have been used as a logical framework for establishing sufficient conditions for cut-admissibility of object logics (OL). However, some logical systems cannot be adequately encoded in LL, the most symptomatic cases being sequent systems for modal logics. In this extended abstract[1], we present a focus linear-nested sequent (LNS) for MMLL (a variant of linear logic with subexponentials), and show that it is possible to establish a cut-admissibility criterion for LNS systems for substructural multi-modal logics.

Introduction. Analytic calculi consist solely of rules that compose the formulas to be proved in a stepwise manner. The best known formalism for proposing analytic proof systems is Gentzen's *sequent calculus.*Unfortunately, sequent systems are not expressive enough for constructing analytic calculi for many modal logics. As a result, many formalisms extending sequent systems have been proposed over the last 30 years, including *hypersequent calculi* ([Avr96]), *nested calculi* ([Brü09]) and *labeled calculi* ([Sim94]).

We study cut-admissibility under the *linear nested system* formalism – LNS ([Lel15]), where a single sequent is replaced with a list of sequents, and the inference rules govern the transfer of formulas between the different sequents. We lift to LNS the method developed by [MP13]. More precisely, we proposed a cut-free focused system for a logic (MMLL) that extends linear logic (LL) [Gir87] with subexponentials featuring different modal behaviors. We also encode different object-level logical systems as theories in MMLL. The proposed encodings are adequate at the highest level and, more interesting, we show that cuts at the object-level can be eliminated by cuts at the MMLL level. Hence, by proving an easy to verify criterion called *cut-coherence*, we obtain for free cut-admissibility results for many modal and substructural logics.

Linear nested systems. A linear nested sequents (LNS) is a finite list of sequents that matches the *history* of a backward proof search in an ordinary sequent calculus [Lel15]. For instance, the modal rules for the axiom K are defined as follows:

$$\frac{\mathcal{G} \mathbin{/\!\!/} \Gamma \vdash \Delta \mathbin{/\!\!/} \cdot \vdash F}{\mathcal{G} \mathbin{/\!\!/} \Gamma \vdash \Delta, \Box F}\ \Box_R \qquad \frac{\mathcal{G} \mathbin{/\!\!/} \Gamma \vdash \Delta \mathbin{/\!\!/} \Gamma', F \vdash \Delta'}{\mathcal{G} \mathbin{/\!\!/} \Gamma, \Box F \vdash \Delta \mathbin{/\!\!/} \Gamma' \vdash \Delta'}\ \Box_L$$

Reading bottom up, while in $\Box_R$ a new nesting/component is created and F is moved there, in $\Box_L$ *exactly one* boxed formula is moved into an existing nesting, losing its modality. Components in a LNS have a tight connection to *worlds* in Kripke-like semantics, so that LNS is an adequate framework for describing alethic modalities. Moreover, information is fragmented

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[1] A full version of this paper, that extends [OPX20], is already under evaluation in *Mathematical Structures in Computer Science.*

Axioms: $\mathsf{K}\ \Box(F \supset G) \supset (\Box F \supset \Box G)$ $\quad \mathsf{D}\ \neg(\Box F \wedge \Box\neg F)$ $\quad \mathsf{T}\ \Box F \supset F$ $\quad \mathsf{4}\ \Box F \supset \Box\Box F$

$$\frac{\Gamma \vdash \Delta /\!\!/ \Sigma, F \vdash \Pi}{\Gamma, \Box F \vdash \Delta /\!\!/ \Sigma \vdash \Pi}\ \Box_L \qquad \frac{\Gamma \vdash \Delta /\!\!/ \cdot \vdash F}{\mathcal{G} /\!\!/ \Gamma \vdash \Delta, \Box F}\ \Box_R \qquad \frac{\Gamma \vdash \Delta /\!\!/ \cdot \vdash \cdot}{\mathcal{G} /\!\!/ \Gamma \vdash \Delta}\ \mathsf{d} \qquad \frac{\mathcal{G} /\!\!/ \Gamma, F \vdash \Delta}{\mathcal{G} /\!\!/ \Gamma, \Box F \vdash \Delta}\ \mathsf{t} \qquad \frac{\Gamma \vdash \Delta /\!\!/ \Sigma, \Box F \vdash \Pi}{\Gamma, \Box F \vdash \Delta /\!\!/ \Sigma \vdash \Pi}\ \mathsf{4}$$

Figure 1: Some modal axioms and their linear nested sequent rules.

into components and rules act locally on formulas and are usually context independent. Hence, the movement of formulas on derivations can be better predicted and controlled.

In this work, besides intuitionistic and classical logics, we are interested in reasoning about linear nested systems for some notable extensions of the normal modal logic K. Fig. 1 presents some modal axioms and the respective linear nested rules. Let $\mathcal{A} = \{\mathsf{T}, \mathsf{4}, \mathsf{D}\}$. Extensions of the logic K are represented by KR, where $\mathsf{R} \subseteq \mathcal{A}$. For instance, $\mathsf{S4} = \mathsf{KT4}$.

Modalities can be combined, giving rise to multi-modal logics. *Simply dependent multi-modal logics* are characterized by a triple $(N, \preccurlyeq, F)$, where N is a denumerable set, $(N, \preccurlyeq)$ is a partial order, and F is a mapping from N to the set $\mathfrak{L}$ of extensions of modal logic K with axioms from the set $\mathcal{A}$. The *logic described by* $(N, \preccurlyeq, F)$ has modalities $\Box_i$ for every $i \in N$, with axioms for the modality i given by the logic $F(i)$ and interaction axioms $\Box_j A \supset \Box_i A$ for every $i, j \in N$ with $i \preccurlyeq j$.

Linear logic with multi-modalities. Classical linear logic (LL, [Gir87]) is a resource conscious logic, in the sense that formulas are consumed when used during proofs, unless marked with the exponential ? (whose dual is !). Formulas marked with ? behave *classically*, *i.e.*, they can be contracted and weakened during proofs. LL connectives include the additive conjunction & and disjunction $\oplus$ and their multiplicative versions $\otimes$ and $⅋$, together with their units.

$\mathsf{LNS_{LL}}$ ([LOP17]) is an end-active, linear nested system for linear logic. In this system, the promotion rule is split into the following local rules:

$$\frac{\vdash \Gamma /\!\!/ \vdash F}{\mathcal{E} /\!\!/ \vdash \Gamma, !F}\ ! \qquad \frac{\vdash \Gamma /\!\!/ \vdash \Delta, ?F}{\vdash \Gamma, ?F /\!\!/ \vdash \Delta}\ ?$$

Observe that no checking must be done in the context in order to apply the ? rule: The only checking is in the ! rule, where $\mathcal{E}$ should be the empty sequent or an empty list of components. Note the similarities between the LNS rules ! and $\Box_R$; and ? and $\mathsf{4}$ in Fig. 1. Indeed, in ([LOP17]) such similarities were exploited in order to propose extensions of $\mathsf{LNS_{LL}}$ with multi-modalities, called *subexponentials*, allowing for different modal behaviors.

Similar to modal connectives, exponentials in LL are not *canonical* ([DJS93]), in the sense that if $i \neq j$ then $!^i F \not\equiv !^j F$ and $?^i F \not\equiv ?^j F$. Intuitively, this means that we can mark the exponentials with *labels* taken from a set $\mathcal{S}$ organized in a pre-order $\preceq$, obtaining (possibly infinitely-many) exponentials ($!^i, ?^i$ for $i \in \mathcal{S}$). Also as in multi-modal systems, the pre-order determines the provability relation: $!^b F$ *implies* $!^a F$ iff $a \preceq b$.

In ([LOP17]) we extended the concept of *simply dependent multimodal logics* to the substructural case, where subexponentials consider not only the structural axioms for contraction ($\mathsf{C}:$ $!^i(F) \multimap !^i F \otimes !^i F$) and weakening ($\mathsf{W}:\ !^i F \multimap 1$) but also the subexponential version of axioms $\{\mathsf{K}, \mathsf{4}, \mathsf{D}, \mathsf{T}\}$: $\mathsf{K}:\ !^i(F \multimap G) \multimap !^i F \multimap !^i G$ $\quad \mathsf{D}:\ !^i F \multimap ?^i F$ $\quad \mathsf{T}:\ !^i F \multimap F$ $\quad \mathsf{4};\ !^i F \multimap !^i !^i F$

This means that $?^i$ can behave classically or not, but also with exponential behaviors different from those in LL. Hence, by assigning different modal axioms one obtains, in a modular way, a class of different substructural modal logics. For instance, subexponentials assuming T allow for dereliction and those assuming $\mathsf{4}$ are persistent (while those assuming only K are not). In fact, substructural KD can be seen as a fragment of elementary linear logic ELL.

Our main goal is to show how this new class of subexponentials can be applied to the problem of characterizing cut-admissibility of object-level logical systems. The first step is to

Structural rules:	$\mathsf{pos}_i : \lfloor A \rfloor^\perp \otimes (?^i \lfloor A \rfloor)$	$\mathsf{neg}_i : \lceil A \rceil^\perp \otimes (?^i \lceil A \rceil)$
Intuitionistic implication:	$\supset_L: \lfloor A \supset B \rfloor^\perp \otimes (\lceil A \rceil \otimes \lfloor B \rfloor)$	$\supset_R: \lceil A \supset B \rceil^\perp \otimes !^{t4}(\lfloor A \rfloor \parr \lceil B \rceil)$
Modal rules:	$\Box_{Li} : \lfloor \Box A \rfloor^\perp \otimes ?^i \lfloor A \rfloor$	$\Box_{Ri} : \lceil \Box A \rceil^\perp \otimes !^i \lceil A \rceil$

Figure 2: Encoding of structural, intuitionistic implication and modal rules.

propose a focused [And92] system for the logic. Below the modal rules of the system:

$$\frac{\vdash \Theta; \cdot \Uparrow \cdot /\!/^i \vdash \cdot; \cdot \Uparrow F}{\vdash \Theta; \cdot \Downarrow\, !^i F}\ !^i \qquad \frac{\vdash \Upsilon; \cdot \Uparrow L}{\vdash \Theta^u; \cdot \Uparrow \cdot /\!/^i \vdash \Upsilon; \cdot \Uparrow L}\ \mathsf{R_r} \qquad \frac{\vdash \Theta; \cdot \Uparrow \cdot /\!/^i \vdash \cdot; \cdot \Uparrow \cdot}{\vdash \Theta; \cdot \Uparrow \cdot}\ \mathsf{D_d} \qquad \frac{\vdash \Theta^u; \cdot \Uparrow F}{\vdash \Theta^u; \cdot \Downarrow\, !^{\mathsf{c}} F}\ !^{\mathsf{c}}$$

$$\frac{\vdash \Theta; \Gamma \Uparrow \cdot /\!/^i \vdash \Upsilon, j+ : F; \cdot \Uparrow L}{\vdash \Theta, j : F; \Gamma \Uparrow \cdot /\!/^i \vdash \Upsilon; \cdot \Uparrow L}\ ?^i{}_{\mathsf{4}} \quad \frac{\vdash \Theta; \cdot \Uparrow \cdot /\!/^i \vdash \Upsilon; \cdot \Uparrow L, F}{\vdash \Theta, j : F; \cdot \Uparrow \cdot /\!/^i \vdash \Upsilon; \cdot \Uparrow L}\ ?^i{}_{\mathsf{kl}} \quad \frac{\vdash \Theta; \cdot \Uparrow \cdot /\!/^i \vdash \Upsilon, \mathsf{c} : F; \cdot \Uparrow L}{\vdash \Theta, j : F; \cdot \Uparrow \cdot /\!/^i \vdash \Upsilon; \cdot \Uparrow L}\ ?^i{}_{\mathsf{ku}}$$

These rules have some interesting characteristics that ease the use of the system and its formalization in Coq (https://github.com/meta-logic/MMLL). Consider a subexponential j. When j features the axiom $\mathsf{4}$, the rules $?^i{}_{\mathsf{kl}}$ (linear K) and $?^i{}_{\mathsf{ku}}$ (unbounded K) cannot be applied. Dually, if the subexponential does not feature $\mathsf{4}$, the rule $?^i{}_{\mathsf{4}}$ is not enabled and the use of $?^i{}_{\mathsf{kl}}$ (resp. $?^i{}_{\mathsf{ku}}$) is only possible if j is linear (resp. unbounded). The rules have also a better control of contraction, thus avoiding the need of guessing the number of times a formula must be copied to the next component. Note that the rule $?^i{}_{\mathsf{4}}$ moves the formula F stored in the context j to the context $j+$ (a unbounded version of j featuring T). This has two immediate effects: The formula F can be copied to yet another component (once it is created) reflecting the behavior of the modal rule $\mathsf{4}$ (persistence); moreover, since the axiom T is present in j, the formula F can be also used in the last component by applying the decision rule. In other words, the rule $?^i_{\mathsf{4}}$ embeds both the behavior of K (moving formulas between components) and also $\mathsf{4}$ (by keeping the modality of the formula). On the other hand, the behavior of K, without $\mathsf{4}$, is specified by the rules $?^i{}_{\mathsf{kl}}$ and $?^i{}_{\mathsf{ku}}$. In the first case, j is linear and then F is not contracted. In the second case, F is placed in the context c, an unbounded subexponential not related to any other subexponential. Hence, F cannot be moved to other components.

We have proved cut-elimination for this system by using five different cut-rules that are mutually eliminated. Such procedure have been mechanized in Coq.

Object logics. We have shown that different LNS systems can be specified as MMLL theories. The encoding of the OL's inference rules is modular and it allows for the specification of multi-modal logics in a uniform way. We have proved that the resulting specifications are *adequate*: an OL sequent S is provable iff the encoding of S is also provable in MMLL.

Roughly, OL *formulas* are specified using the meta-level (MMLL) predicates $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$, that identify the occurrence of such formulas on the left and on the right side of the sequent respectively. Hence, OL sequents of the form $B_1, \ldots, B_n \vdash C_1, \ldots, C_m$, $n, m \geq 0$, are specified as the multiset of atomic MMLL formulas $\lfloor B_1 \rfloor, \ldots, \lfloor B_n \rfloor, \lceil C_1 \rceil, \ldots, \lceil C_m \rceil$.

Inference rules of the OL are specified as rewriting clauses that replace the principal formula in the conclusion of the rule by the active formulas in the premises. The LL connectives indicate how these OL formulas are connected: contexts are copied ($\&$) or split ($\otimes$), in different inference rules ($\oplus$) or in the same sequent ($\parr$). Here some examples for the classical logic connectives:

$\wedge_L : \lfloor A \wedge B \rfloor^\perp \otimes (\lfloor A \rfloor \oplus \lfloor B \rfloor)$	$\wedge_R : \lceil A \wedge B \rceil^\perp \otimes (\lceil A \rceil \,\&\, \lceil B \rceil)$	$\mathtt{f}_L : \lfloor \mathtt{f} \rfloor^\perp \otimes \top$
$\rightarrow_L: \lfloor A \rightarrow B \rfloor^\perp \otimes (\lceil A \rceil \otimes \lfloor B \rfloor)$	$\rightarrow_R: \lceil A \rightarrow B \rceil^\perp \otimes (\lfloor A \rfloor \parr \lceil B \rceil)$	$\mathsf{init} : \lfloor A \rfloor^\perp \otimes \lceil A \rceil^\perp$

In the intuitionistic system $\mathsf{LNS_I}$ [LOP17], the rule $\supset_R$ creates a new component while `lift` moves formulas across components. Such behavior can be specified with the unbounded subexponential $t4$ featuring K, T and $\mathsf{4}$ as in Fig. 2. This figure also shows the (parameterized) clauses specifying the rules for box. As expected, the modalities of the subexponential i are determined by the modal behavior of the encoded modality $\Box_i$.

It is worth noticing the *modularity* of the encodings: all the modal systems have *exactly* the same encoding, only differing on the meta-level modality. This is a direct consequence of locality, granted by LNS. This also opens the possibility of being able to adequately encode a larger class of modal systems. For instance, if we are considering a (modal) substructural logic where formulas not necessarily behave classically, it suffices to remove the clauses pos and/or neg accordingly.

Cut-elimination for object logics. We showed that an easy-to-check criterium, called *cut-coherence* implies that cuts at the object-level can be eliminated by cuts at the meta-level.

Consider the (multiplicative) OL cut rule specified as the clause $\mathsf{cut} = \exists F.(\lfloor F \rfloor \otimes \lceil F \rceil)$. Cut-coherence is the property that allows us to show the duality, at the meta-level, of the predicates $\lfloor F \rfloor$ and $\lceil F \rceil$. More precisely, let $\mathcal{C}$ be the set of connectives of the OL $\mathcal{L}$. The *encoding* of $\mathcal{L}$ as an MMLL theory is a pair of functions $\mathtt{B}\lfloor \cdot \rfloor$ and $\mathtt{B}\lceil \cdot \rceil$ from $\mathcal{C}$ to MMLL of the form $\mathtt{E}\lfloor \star \rfloor = \exists F_1, ..., F_n.(\lfloor \star(F_1, ..., F_n) \rfloor^\perp \otimes \mathtt{B}\lfloor \star \rfloor)$ $\qquad \mathtt{E}\lceil \star \rceil = \exists F_1, ..., F_n.(\lceil \star(F_1, ..., F_n) \rceil^\perp \otimes \mathtt{B}\lceil \star \rceil)$. We say that the resulting MMLL theory is *cut-coherent* if, for each connective $\star \in \mathcal{C}$, and $F = \star(F_1, ..., F_n)$, the following sequent is provable $\vdash \omega : \mathsf{cut}; \Uparrow \forall F_1, ..., F_n.((\mathtt{B}\lfloor \star \rfloor)^\perp \parr (\mathtt{B}\lceil \star \rceil)^\perp)$.

All the encodings we have proposed for substructural modal logics based on multiplicative-additive linear logic with different modalities extending K are cut-coherent. Then, for all these encodings, the following result can be applied.

Theorem: Cut-coherence. Let $\mathcal{T}_{\mathcal{L}}$ be the theory of a given OL $\mathcal{L}$, and let Ψ be a multiset and Θ a subexponential context containing only atoms of the form $\lceil \cdot \rceil$ and $\lfloor \cdot \rfloor$. The sequent $\vdash \omega : \{\mathcal{T}_{\mathcal{L}}, \mathsf{cut}\}, \Theta; \Psi \Uparrow \cdot$ is provable iff $\vdash \omega : \mathcal{T}_{\mathcal{L}}, \Theta; \Psi \Uparrow \cdot$ is provable.

As future work, it would be interesting to analyze the case of non-normal modal logics ([LP19]), as well as to explore the failure cases.

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