Intuitionistic modal algebras and twist representations

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A *modal Heyting algebra* is obtained by enriching a Heyting algebra $\langle H; \wedge, \vee, \to, 0, 1\rangle$ with a unary modal operator $\Box$ satisfying the following identity:

$$x \to \Box y = \Box x \to \Box y.$$

Such an operator is also known in the literature as a *nucleus*, or a *multiplicative closure operator*. Many natural constructions give rise to nuclei. For instance, having fixed an element $a \in H$ of a Heyting algebra, we can obtain a nucleus by setting either $\Box x := a \to x$ or $\Box x := a \vee x$, or $\Box x := (x \to a) \to a$. So, in particular, the identity map, the constant map $x \mapsto 1$ and the double negation map also define nuclei (see [8, 1] for further examples).

The class of modal Heyting algebras (and some of its subreducts) has been studied since the 1970s, usually within the framework of topology and sheaf theory [8, 9, 3, 2]. A more recent paper [5] proposed a logic based on modal Heyting algebras (called *Lax Logic*) as a tool in the formal verification of computer hardware. Even more recently, another connection between modal Heyting algebras and logic emerged within the study of the algebraic semantics of *quasi-Nelson logic* [16, 15]. The latter may be viewed as a common generalization of both intuitionistic logic and *Nelson's constructive logic with strong negation* [10] obtained by deleting the double negation law.

As shown in [15, 12, 11], there exists a formal relation between the algebraic counterpart of quasi-Nelson logic and the class of modal Heyting algebras which parallels the well-known connection between *Nelson algebras* and Heyting algebras (see e.g. [17]). This relation – which, as we shall see, concerns the algebras in the full language as well as some of their subreducts – provides, in our view, further motivation for the study of modal Heyting algebras from a logical as well as an algebraic point of view. It is interesting to note that, with the notable exception of [1], studies of this kind are scant in the literature – perhaps owing to the mainly topological interest in this class of algebras? The purpose of the present contribution is to fill in this gap, at least partly, and at the same time to draw attention to certain subreducts of modal Heyting algebras whose interest is motivated by recent developments in the theory of quasi-Nelson logic.

Since a modal Heyting algebra is usually presented in the language $\{\wedge, \vee, \to, \Box, 0, 1\}$, fragments that appear to be of natural interest (from a logico-algebraic perspective) are, for instance, the implication-free one $\{\wedge, \vee, \Box\}$ – perhaps enriched with the lattice bounds 0 and 1 – and the implicational one $\{\to, \Box\}$. The former, whose models are distributive lattices enriched with a modal operator, is in fact the main object of [1], while the latter – whose models are *Hilbert algebras*, the algebraic counterpart of the purely implicational fragment of intuitionistic logic, expanded with a modal operator – was studied, mainly from a topological perspective, as far back as in [8], and as recently as in [4]. Other less obvious but, in our opinion, also interesting classes of algebras emerged in the course of our recent investigations on quasi-Nelson logic

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and its algebraic counterpart, the variety of *quasi-Nelson algebras*. An interest in these classes of algebras, however, can also be motivated within the limits of the traditional framework of modal Heyting algebras, as explained below.

A well-known fact on modal Heyting algebras [8, Thm. 2.12] is that, for every such algebra $\mathbf{H} = \langle H; \wedge, \vee, \to, \Box, 0, 1 \rangle$, the set $H_\Box := \{a \in H : a = \Box a\}$ of fixpoints of the $\Box$ operator can itself be endowed with a modal Heyting algebra structure by defining, for every n-ary algebraic operation $f \in \{\wedge, \vee, \to, \Box, 0, 1\}$, the operation $f_\Box$ given, for all $a_1, \ldots, a_n \in H_\Box$, by $f_\Box(a_1, \ldots, a_n) := \Box f(a_1, \ldots, a_n)$.

Denoting this algebra by $\mathbf{H}_\Box$, we observe that, the universe $H_\Box$ can equivalently be defined as the nucleus image $\{\Box a : a \in H\}$ of $\mathbf{H}$. While $\mathbf{H}_\Box$ is indeed a modal Heyting algebra, it is a very special one on which the $\Box$ operator is the identity map. This very fact, in turn, is essential in ensuring that $\mathbf{H}_\Box$ has a Heyting algebra reduct; for instance we have, for all $a, b \in H_\Box$,

$$a \wedge_\Box b = \Box(a \wedge b) = \Box a \wedge \Box b = a \wedge b$$

guaranteeing that $\wedge_\Box$ is a meet semilattice operation on $H_\Box$. A similar reasoning applies to the other operations, although the join $\vee_\Box$ (computed in $\mathbf{H}_\Box$) does not coincide with the join $\vee$ (computed in $\mathbf{H}$), i.e. $\mathbf{H}_\Box$ is not a subalgebra of $\mathbf{H}$. This construction is easily seen to be a generalization of Glivenko's result relating Heyting and Boolean algebras (the latter corresponding to the case where $\Box x = \neg\neg x$).

Thus, although nothing prevents one from considering each operation $f_\Box$ as defined on the whole universe H, in general $\wedge_\Box$ and $\vee_\Box$ will not be semilattice operations on H, and $\to_\Box$ will not be a Heyting (i.e. a relative pseudo-complement) implication on H (on the other hand, we always have $\Box_\Box = \Box$ and $1_\Box = 1$). By definition, these new operations will be generalizations of the intuitionistic ones, which can be retrieved by requiring $\Box$ to be the identity map on H. In this respect natural questions to ask are, in our opinion, (1) which properties each generalized operation $f_\Box$ retains, and (2) whether some particular choice of $f_\Box$ has any independent interest that may justify further study.

A first answer to the latter question may be sought within the theory of quasi-Nelson logic. Indeed, as shown in the papers [15, 12, 11, 13], some of the above-defined operations of type $f_\Box$ naturally arise within the study of fragments of the quasi-Nelson language. From this standpoint, it is also interesting to observe that the classes of algebras one obtains through the *twist representation* (see below) combine the original Heyting operations with the new ones. Thus, for instance, one of the classes of algebras arising in this way retains the original meet semilattice operation (and the lattice bounds) while replacing the Heyting implication with a generalized counterpart: that is, we are looking at the $\{\wedge, \to_\Box, 0, 1\}$-subreducts of modal Heyting algebras. We stress that these new algebras are not the result of an arbitrary choice of operations, but arise as twist factors in the representation of subreducts of quasi-Nelson algebras, as we now proceed to explain.

A *quasi-Nelson algebra* may be defined as a commutative integral bounded residuated lattice (see e.g. [6] for formal definitions of these terms) $\mathbf{A} = \langle A; \sqcap, \sqcup, *, \Rightarrow, \bot \rangle$ that (upon letting ${\sim} x := x \Rightarrow \bot$) satisfies the *Nelson identity*: $(x \Rightarrow (x \Rightarrow y)) \sqcap ({\sim} y \Rightarrow ({\sim} y \Rightarrow {\sim} x)) = x \Rightarrow y$.

Quasi-Nelson algebras arise as the algebraic counterpart of quasi-Nelson logic, which can be viewed either as a generalization (i.e. a weakening) common to Nelson's constructive logic with strong negation and to intuitionistic logic, or as the extension (i.e. strengthening) of the well-known substructural logic FL_{ew} (the *Full Lambek Calculus with Exchange and Weakening*) by the *Nelson axiom*:

$$((x \Rightarrow (x \Rightarrow y)) \sqcap ({\sim} y \Rightarrow ({\sim} y \Rightarrow {\sim} x))) \Rightarrow (x \Rightarrow y).$$

We refer to [16] for further details on quasi-Nelson logic, as well as for other equivalent characterizations of the variety of quasi-Nelson algebras (which can e.g. also be obtained as the class of $(0,1)$-*congruence orderable* commutative integral bounded residuated lattices).

Formally, every Heyting algebra may be viewed as a quasi-Nelson algebra (on which $\wedge = *$, $\vee = \sqcup$, $\to = \Rightarrow$ and $0 = \bot$) and, as noted earlier, the double negation map defines a modal operator on every Heyting algebra $\mathbf{H}$. If we replace $\mathbf{H}$ by a quasi-Nelson algebra $\mathbf{A}$, then the double negation map need not define a nucleus on $\mathbf{A}$, but can be used to obtain one on a special quotient $H(\mathbf{A})$, which is the (Heyting) algebra canonically associated to each quasi-Nelson algebra $\mathbf{A}$ via the twist construction.

Given a quasi-Nelson algebra $\mathbf{A}$, consider the map given, for all $a \in A$, by $a \mapsto a * a$. The kernel θ of this map is a congruence of the reduct $\langle A; \sqcap, \sqcup, * \rangle$ which is also compatible with the double negation operation and with the *weak implication* $\Rightarrow^2$ given by $x \Rightarrow^2 y := x \Rightarrow (x \Rightarrow y)$. Letting $\Box(x/\theta) := {\sim}{\sim} x/\theta$, we thus have a quotient algebra $H(\mathbf{A}) = \langle A/\theta; \sqcap, \sqcup, \Rightarrow^2, \Box, \bot \rangle$, which is a modal Heyting algebra (where $* = \sqcap$). Moreover, $\mathbf{A}$ embeds into a *twist-algebra* over $H(\mathbf{A})$, defined as follows.

Given a a modal Heyting algebra $\mathbf{H} = \langle H; \wedge, \vee, \to, \Box, 0, 1 \rangle$, define the algebra $\mathbf{H}^{\bowtie} = \langle H^{\bowtie}; \sqcap, \sqcup, *, \Rightarrow, \bot \rangle$ with universe $H^{\bowtie} := \{\langle a_1, a_2 \rangle \in H \times H_{\Box} : a_1 \wedge a_2 = 0\}$ and operations given, for all $\langle a_1, a_2 \rangle, \langle b_1, b_2 \rangle \in H \times H$, by:

$$
\begin{aligned}
\bot &:= \langle 0, 1 \rangle \\
\langle a_1, a_2 \rangle * \langle b_1, b_2 \rangle &= \langle a_1 \wedge b_1, \Box((a_1 \to b_2) \wedge (b_1 \to a_2)) \rangle \\
\langle a_1, a_2 \rangle \sqcap \langle b_1, b_2 \rangle &:= \langle a_1 \wedge b_1, \Box(a_2 \vee b_2) \rangle \\
\langle a_1, a_2 \rangle \sqcup \langle b_1, b_2 \rangle &:= \langle a_1 \vee b_1, \Box(a_2 \wedge b_2) \rangle \\
\langle a_1, a_2 \rangle \Rightarrow \langle b_1, b_2 \rangle &:= \langle (a_1 \to b_1) \wedge (b_2 \to a_2), \Box(a_1 \wedge b_2) \rangle.
\end{aligned}
$$

A *quasi-Nelson twist-algebra over* $\mathbf{H}$ is any subalgebra $\mathbf{A} \leq \mathbf{H}^{\bowtie}$ satisfying $\pi_1[A] = H$.

The *twist representation theorem* says that every quasi-Nelson algebra $\mathbf{A}$ embeds into the twist-algebra $(H(\mathbf{A}))^{\bowtie}$ through the map given by $a \mapsto \langle a/\theta, {\sim} a/\theta \rangle$ [16].

The previous definition suggests that certain term operations of the language of modal Heyting algebras may be of particular interest in the study of fragments of the quasi-Nelson language. Consider, for instance, the monoid operation $(*)$. In order to define it, on a quasi-Nelson algebra $\mathbf{A} \leq \mathbf{H}^{\bowtie}$, we need two operations on $\mathbf{H}$: the semilattice operation $\wedge$ (for the first component) and, for the second component, an implication-like operation (let us denote it by $\rightharpoonup$) which can be given by $x \rightharpoonup y := x \to \Box y$. The latter claim may not be obvious, but using the properties of the twist construction and the modal operation, it is not hard to verify the following equalities:

$$
\begin{aligned}
\Box((a_1 \to b_2) \wedge (b_1 \to a_2)) &= \Box((a_1 \to \Box b_2) \wedge (b_1 \to \Box a_2)) \\
&= \Box(a_1 \to \Box b_2) \wedge \Box(b_1 \to \Box a_2) \\
&= (a_1 \to \Box b_2) \wedge (b_1 \to \Box a_2) \\
&= (a_1 \rightharpoonup b_2) \wedge (b_1 \rightharpoonup a_2).
\end{aligned}
$$

These observations led to the introduction of the class of algebras dubbed $\rightharpoonup$-*semilattices* in [13], where it is shown in particular that the $\{*, \sim\}$-subreducts of quasi-Nelson algebras are precisely the algebras representable as twist-algebras over $\rightharpoonup$-semilattices. Similar considerations motivated the introduction of other term operations of the language of modal Heyting algebras, such as the following: $x \odot y := \Box(x \wedge y)$ and $x \oplus y := \Box(x \vee y)$. As shown in [13], the corresponding classes of modal algebras allow us to establish twist representations for (respectively) the classes of $\{\Rightarrow^2, \sim\}$-subreducts and of $\{\wedge, *, \Rightarrow, \sim\}$-subreducts of quasi-Nelson algebras. Other subreducts may be obtained by adding a modal operator to more traditional classes of intuitionistic algebras, such as implicative semilattices (corresponding to the $\{*, \Rightarrow, \sim\}$-subreducts of quasi-Nelson algebras), distributive lattices (corresponding to the $\{\wedge, \vee, \sim\}$-subreducts studied in [14]) and pseudo-complemented lattices (corresponding to the "two-negations" subreducts studied in [12]).

The previous considerations suggest the above-mentioned classes of modal algebras as mathematical objects that may be of interest both in themselves and in relation to the study of non-classical logics, in particular Nelson's logics[1]. The aim of the present contribution is to improve our understanding of these classes of algebras from an algebraic as well as a topological point of view.

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