Logical Approximations of Qualitative Probability

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Comparative structures provide a natural bridge between the logical and probabilistic representation of uncertainty, of relevance both for the foundations of probability and statistics [5], and AI[6]. Formally, a comparative structure is a pair $(\mathcal{A}, \preceq)$ where $\mathcal{A}$ is a boolean algebra and $\preceq$ is interpreted as a *qualitative probability (relation)* on $\mathcal{A}$, i.e. we write $\theta \preceq \phi$ to say that θ is no-more-probable-than ϕ, for any $\theta, \phi \in \mathcal{A}$. The relations $\theta \approx \phi$ and $\theta \prec \phi$ are defined from $\preceq$ as usual.

Definition 1 (Comparative structure)**.** $(\mathcal{A}, \preceq)$ *is a* comparative structure *if*

1. $\preceq$ *is a total preorder over* $\mathcal{A}$*;*
2. $\bot \prec \top$*;*
3. *if* $\alpha \sqsubseteq \beta$ *then* $\alpha \preceq \beta$ *and*
4. *if* $\alpha \wedge \gamma = \bot$ *and* $\beta \wedge \gamma = \bot$ *then*

$$\alpha \preceq \beta \text{ if and only if } \alpha \vee \gamma \preceq \beta \vee \gamma.$$

Recall that by $\sqsubseteq$ we denote the lattice order of the Boolean algebra, to be distinguished by $\preceq$. The definition above is essentially due to [4] who introduced condition *4.* as the qualitative counterpart of additivity.

De Finetti thought of comparative structures as the logical core of uncertain reasoning, and conjectured that they would be representable by usual probability functions. Let us recall the following.

Definition 2 ((Almost) Representability)**.** *A comparative structure* $(\mathcal{A}, \preceq)$ *is said to be :*

- representable *if there exists a finitely additive probability* P *such that* $\alpha \preceq \beta$ *iff* $P(\alpha) \leq P(\beta)$*;*
- almost representable *if there exists a finitely additive probability* P *such that* $\alpha \preceq \beta$ *implies* $P(\alpha) \leq P(\beta)$.

However, even almost representability fails to hold for general comparative structures, as shown in 1959 by [8]. Since then, various authors have proposed additional suitable axioms for establishing (almost) representability, e.g. [10, 9, 11, 7]. All these approaches present however drawbacks, in that they make make strong idealizing assumptions, e.g.:

- $\preceq$ has to contain the classical relation $\sqsubseteq$. But finding out whether $\sqsubseteq$ holds is non-feasible, according to standard assumptions in computational complexity.
- The axioms that need to be added to Definition 1, to obtain representability either postulate [10] that there are arbitrarily fine-grained events to be compared, or impose conditions which are hard to interpret intuitively [9, 11]

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In this work, we address these problems, by developing a sequence of comparative structures, which are meant to be *approximations* of almost representable comparative structures. While each structure in the sequence is not by itself representable, we attain the result only in the limit, provided that the sequence satisfies certain conditions.

Our framework is crucially based on Depth-Bounded Boolean Logics [3, 2], instead of classical logic. These logics are centered around the idea of limiting the applications of the bivalence principle, which holds unboundedly for classical logic. In natural deduction-style the principle may be presented as follows:

$$\dfrac{\begin{matrix}[\varphi] \\ \vdots \\ \psi\end{matrix} \qquad \begin{matrix}[\neg\varphi] \\ \vdots \\ \psi\end{matrix}}{\psi}\ (\mathrm{PB})$$

This means that to infer the formula ψ, it suffices to infer it both under the assumption that φ is the case and under the assumption that $\neg\varphi$ is the case. The square brackets around the formulas φ and $\neg\varphi$ signal that those are pieces of information assumed for the sake of deriving ψ, but not actually held true (they are *discharged*, in natural deduction terminology). We call this type of information *hypothetical*, in contrast to the *actual* information which an agent may hold as her premises.

[3] introduces a logic $\vdash_0$, which does not allow any manipulation of hypothetical information, i.e. any application of PB, and is defined proof-theoretically by a core set of introduction and elmination rules (Intelim Rules [3]), for each connective, both when occurring positively (as the main connective of a formula) and negatively (in the scope of a negation). The family of Depth-Bounded Boolean Logics $\{\vdash_k\}_{k\in\mathbb{N}}$ is then characterized, for $k > 0$, by allowing, in addition to the rules of $\vdash_0$, at most k nested applications of PB.

Results in [3] show that:

- $\vdash_0 \subset \vdash_1 \subset \cdots \subset \vdash_k \subset \cdots$, so the depth-bounded consequence relations form a hierarchy;
- $\lim_{k\to\infty} \vdash_k = \vdash$, i.e. at the limit, the hierarchy of depth-bounded boolean logics coincides with classical logic;
- for each k, $\vdash_k$ has a polynomial decision procedure.

These properties make these logics a suitable starting point for addressing the problems of comparative structures discussed above. In this work, we shall: define a sequence of bounded comparative structures, based on the sequence of depth-bounded boolean logics; identify the conditions under which the bounded comparative structures are asymptotically representable by a probability measure and, conversely, those conditions under which a representable qualitative probability structure can be approximated.

If time allows, we will also discuss current work in progress, in two directions: determining that resulting approximating comparative structures are tractable, along the lines of work done in [1] for the quantitative case, and devising a decision-theoretic framework, on the model of Savage's [10] which grounds our bounded comparative structures on a corresponding notion of bounded preferences between acts.

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