Relevant Reasoners in a Classical World

Igor Sedlár[1] and Pietro Vigiani[2]

[1] Czech Academy of Sciences, Institute of Computer Science, Prague, The Czech Republic
sedlar@cs.cas.cz

[2] Scuola Normale Superiore, Department of Philosophy, Pisa, Italy
pietro.vigiani@sns.it

In this paper we provide a framework for epistemic logic based on relevant modal logic aimed at avoiding the *logical omniscience* problem. In particular, we will be interested in the following instances of the problem, where $\Box$ models belief and $n \geq 0$:

$$\frac{\varphi_1 \wedge \cdots \wedge \varphi_n \to \psi}{\Box\varphi_1 \wedge \cdots \wedge \Box\varphi_n \to \Box\psi} \quad \textit{Conjunctive Regularity} \quad \text{(C-Reg)}$$

$$\frac{\varphi_1 \to (\ldots(\varphi_n \to \psi)\ldots)}{\Box\varphi_1 \to (\ldots(\Box\varphi_n \to \Box\psi)\ldots)} \quad \textit{Implicative Regularity} \quad \text{(I-Reg)}$$

Famously, standard relational semantics for normal modal logics validate both. On the other hand, relational semantics for *relevant* modal logic [3] avoids (I-Reg) and the special case of (C-Reg) for $n = 0$, Necessitation. The difference between (C-Reg) and (I-Reg) expresses the assumption that while beliefs of agents are represented as "automatically" closed under conjunction introduction, they are not seen as closed under implication elimination. As Sequoiah-Grayson [7] points out, this can be understood as meaning that while agents are assumed to automatically *aggregate* their beliefs, they are not assumed to automatically *combine* them.

In the relational semantics for relevant modal logics, validity is defined in terms of a set of *logical states*, but the failure of (I-Reg) is made possible by allowing the modal accessibility relation to reach out of the set of logical states. This is a feature the relevant semantics has in common with the so-called *non-normal states* approaches to the logical omniscience problem [4, 5, 8]. In these approaches, however, the set of normal states consists of classical possible worlds. The logic generated by these semantics extends classical propositional logic with epistemic modalities that are not closed under inference rules of classical propositional logic.

It makes sense to assume, though, that epistemic modalities are closed under *some* logic. More specifically, the requirement of a *relevant* connection between a piece of information and a conclusion agents draw on its basis makes some form of relevant logic a natural candidate. It has been argued, for instance, that processing an input φ in a context yields 'a contextual implication, a conclusion [ψ] deducible from the input and the context together, but from neither input nor context alone' [9]. As noted in [1], such informational interpretation of relevance is embodied in Routley and Meyer's relational semantics for relevant logics, in particular in the ternary relation interpreting implication.

In classical epistemic logic, it is sufficient for $\Box\varphi_1 \wedge \ldots \wedge \Box\varphi_n \to \Box\psi$ to be valid that ψ is classically implied by $\varphi_1 \wedge \ldots \wedge \varphi_n$. On the relevant criterion, the classical validity of the salient implication should not be sufficient. However, its *relevant* validity should.

In this presentation, we outline a framework for relevant epistemic logic based on these ideas. Our framework models agents as *relevant reasoners in a classical world*: the agent reasons in accordance with a relevant modal logic, but the propositional fragment of our logic is classical.

More specifically, we consider a wide range of relevant modal logics, extending the following system $\mathsf{BM.C}$, based on the basic system considered in [3]:

(a1)	$p \to p$	(a7)	$q \to (p \vee q)$
(a2)	$\neg(p \wedge q) \to (\neg p \vee \neg q)$	(a8)	$((p \to q) \wedge (p \to r)) \to (p \to (q \wedge r))$
(a3)	$(\neg p \wedge \neg q) \to \neg(p \vee q)$	(a9)	$((p \to r) \wedge (q \to r)) \to ((p \vee q) \to r)$
(a4)	$(p \wedge q) \to p$	(a10)	$(p \wedge (q \vee r)) \to ((p \wedge q) \vee (p \wedge r))$
(a5)	$(p \wedge q) \to q$	(a11)	$(\Box p \wedge \Box q) \to \Box(p \wedge q)$
(a6)	$p \to (p \vee q)$	(a12)	$(\Box_L p \wedge \Box_L q) \to \Box_L(p \wedge q)$

plus the rules of Uniform substitution (US) and Modus ponens (MP) and

$$\text{(Adj)}\ \frac{\varphi \qquad \psi}{\varphi \wedge \psi} \qquad \text{(Aff)}\ \frac{\varphi' \to \varphi \qquad \psi \to \psi'}{(\varphi \to \psi) \to (\varphi' \to \psi')}$$

$$\text{(Con)}\ \frac{\varphi \to \psi}{\neg\psi \to \neg\varphi} \qquad (\Box_L\text{-Mon})\ \frac{\varphi \to \psi}{\Box_L\varphi \to \Box_L\psi}$$

$$(\Box\text{-Mon})\ \frac{\varphi \to \psi}{\Box\varphi \to \Box\psi}$$

Then, for each relevant modal logic L, extending $\mathsf{BM.C}$ with axioms/rules corresponding to stronger propositional and modal properties of the agent, we develop a "classical" modal logic CL. The key feature of our framework, connecting L and CL, is the *relevant reasoning (meta)rule*

$$\frac{\vdash_{\mathsf{L}} \varphi_1 \wedge \ldots \wedge \varphi_n \to \psi}{\vdash_{\mathsf{CL}} \Box\varphi_1 \wedge \ldots \wedge \Box\varphi_n \to \Box\psi} \tag{RR}$$

for $n \geq 1$. In order to obtain closure under (RR), we introduce the auxiliary modal operator $\Box_L$ "expressing" provability in L in the sense that

$$\vdash_{\mathsf{L}} \varphi \iff \vdash_{\mathsf{CL}} \Box_L\varphi \tag{LCL}$$

Each L is closed under (C-Reg) and we will prove that CL proves $\varphi \to \psi$ if it proves $\Box_L(\varphi \to \psi)$. Hence, closure under (RR).

In order to ensure that the propositional fragment of CL is classical propositional logic, CPC, we modify the standard relational semantics of relevant modal logic. In our semantics based on so-called W-models, validity in a model is defined as satisfaction throughout a set of designated states that, as far as propositional connectives are concerned, behave like classical possible worlds.

We stress that while (RR) is satisfied, the standard logical omniscience problem is avoided in our framework since CL is generally *not* closed under (C-Reg) nor under Necessitation. This follows from the fact that while validity is defined as satisfaction in all standard states (in our case, possible worlds), the epistemic accessibility relation Q may connect standard states with non-standard states.

Definition 1. *A* bounded *frame is a relevant modal frame* $(S, \leq, R, *, Q, Q_L)$ *where* $R \subseteq S^3$ *is downward (upward) monotone in its first and second (third) argument,* $* : S \to S$ *is anti-monotonic and* $Q, Q_L \subseteq S^2$ *are downward (upward) monotone in their first (second) argument. Moreover,* $(S, \leq)$ *is a bounded poset, i.e. there are elements* $0, 1 \in S$ *such that for all* $s \in S$ $0 \leq s \leq 1$*, such that for all* $s, t \in S$*, the following are satisfied* ($Q_{(L)} \in \{Q, Q_L\}$)*:*

$$1^* = 0 \text{ and } 0^* = 1 \tag{1}$$

$$Q_{(L)}00 \tag{2}$$
$$Q_{(L)}1s \Rightarrow s = 1 \tag{3}$$
$$R010 \tag{4}$$
$$R1st \Rightarrow (s = 0 \text{ or } t = 1) \tag{5}$$

Relevant modal frames are a variant of frames as defined by Fuhrmann [3], but with a binary relation Q_L instead of the set of logical states L. The definition of bounded frames is taken from Seki [6].

Definition 2. *A W-*frame *is a structure $\boldsymbol{F} = (F, W)$ where F is a bounded frame, $W \subseteq S$ is a set of possible worlds, i.e. the following conditions are satisfied:*

$$w^* = w \tag{6}$$
$$Rwww \tag{7}$$
$$Rwst \Rightarrow (s = 0 \text{ or } w \leq t) \tag{8}$$
$$Rwst \Rightarrow (t = 1 \text{ or } s \leq w^*) \tag{9}$$
$$(\forall w \in W)(\forall s, t, u)(Q_L wu \ \& \ Rust \Rightarrow s \leq t) \tag{10}$$
$$(\forall s)(\exists w \in W)(\exists u)(Q_L wu \ \& \ Russ) \tag{11}$$

*A W-*model *based on $\boldsymbol{F}$ is $\boldsymbol{M} = (\boldsymbol{F}, V)$ where $V : Pr \to S(\uparrow)$, the set of upward closed subsets of S, such that $1 \in V(p)$ for all p and $0 \notin V(p)$ for all $p \in Pr$.*

Conditions (10)-(11) enable W-frames to simulate validity in relevant modal models. In W-frames, the set of states $Q_L(W) = \{u \mid \exists w(w \in W \ \& \ Q_L wu)\}$ "plays the role" of the set of logical states. For each W-frame F, we define the following operations on 2^S:

$$X \wedge^F Y = X \cap Y \qquad X \vee^F Y = X \cup Y$$
$$X \circ^F Y = \{u \mid \exists s, t(s \in X \ \& \ t \in Y \ \& \ Rstu)\}$$
$$X \to^F Y = \{s \mid \{s\} \circ^F X \subseteq Y\} \qquad \neg^F X = \{s \mid s^* \notin X\}$$
$$\Box^F X = \{s \mid \forall t(Qst \Rightarrow t \in X)\} \qquad \Box^F_L X = \{s \mid \forall t(Q_L st \Rightarrow t \in X)\}$$

and, for each W-model $\boldsymbol{M}$, the $\boldsymbol{M}$- *interpretation* $[\![\,]\!]_{\boldsymbol{M}}$ as a function $[\![\,]\!]_{\boldsymbol{M}} : Fm_{\mathcal{L}} \to S(\uparrow)$ such that $[\![p]\!]_{\boldsymbol{M}} = V(p)$ and

$$[\![c(\varphi_1, \ldots, \varphi_n)]\!]_{\boldsymbol{M}} = c^F\left([\![\varphi_1]\!]_{\boldsymbol{M}}, \ldots, [\![\varphi_n]\!]_{\boldsymbol{M}}\right)$$

for all $c \in \{\wedge, \vee, \to, \neg, \Box, \Box_L\}$. Crucially, a formula φ is valid in a class of W-frames iff it is valid in each W-model based on a W-frame belonging to the class, i.e. iff $W \subseteq [\![\varphi]\!]_{\boldsymbol{M}}$.

Given that validity is defined with respect to a special subset of situations, representing possible worlds, for all W-models $\boldsymbol{M}$ propositional formulas behave classically when interpreted at worlds $w \in W$. That is, we can prove that:

- $(\boldsymbol{M}, w) \models \neg\varphi$ iff $(\boldsymbol{M}, w) \not\models \varphi$
- $(\boldsymbol{M}, w) \models \varphi \to \psi$ iff $(\boldsymbol{M}, w) \not\models \varphi$ or $(\boldsymbol{M}, w) \models \psi$.

Definition 3. *For all relevant modal logics* L*, we define* CL *as the axiom system comprising*

1. CPC *with (MP) and (US) where substitutions are functions from Pr to $Fm_{\mathcal{L}}$;*

2. *for all axioms* φ *of* L*, an axiom* $\Box_L\varphi$*, and for all inference rules* $\dfrac{\varphi_1 \ldots \varphi_n}{\psi}$ *of* L*, the rule* $\dfrac{\Box_L\varphi \ldots \Box_L\varphi_n}{\Box_L\psi}$*;*

3. *The Bridge Rule (BR)* $\dfrac{\Box_L(\varphi \to \psi)}{\varphi \to \psi}$.

The fact that each CL is closed under (RR) for all $n > 0$ is established as follows. If $\vdash_{\mathsf{L}} \bigwedge_{i \leq n} \varphi_i \to \psi$, then $\vdash_{\mathsf{L}} \bigwedge_{i \leq n} \Box\varphi_i \to \Box\psi$ using monotonicity and regularity of $\Box$ in L, and so $\vdash_{\mathsf{CL}} \Box_L\big(\bigwedge_{i \leq n} \Box\varphi_i \to \Box\psi\big)$ by (LCL). But then $\vdash_{\mathsf{CL}} \bigwedge_{i \leq n} \Box\varphi_i \to \Box\psi$ follows using (BR).

Our main technical result is a general completeness theorem for CL with respect to W-models.

Theorem 1. *For any logic* L *and* W*-model* $\boldsymbol{M}$*,* $\vdash_{\mathsf{CL}} \varphi \Leftrightarrow W \subseteq [\![\varphi]\!]_{\boldsymbol{M}}$*.*

After proving the completeness theorem, we will discuss the following generalization of our framework. Each logic L considered in [3] contains the axiom (C) and is closed under (C-Reg). In contrast to [7], a case can be made against conjunctive regularity for $\wedge$, arguing that agents' beliefs tend to come in non-interacting clusters, or frames of mind [2], and therefore belief aggregation is not automatic. A natural generalisation of the present framework explores a neighborhood semantics for the epistemic modality, where crucially the collection of sets in the neighborhood of a state need not be closed under intersection. A relevant modal logic L based on neighborhood semantics then would have the congruence rule

$$\frac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi}\,, \tag{Con}$$

as its only distinctively modal principle. We will outline how our completeness result generalizes to the neighborhood setting.

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