Strong standard completeness for S5-modal Łukasiewicz [2]

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Abstract

We study the S5-modal expansion of the logic based on the Łukasiewicz t-norm. We exhibit an infinitary propositional calculus and show that it is strongly complete with respect to this logic. These results are derived from properties of monadic MV-algebras: functional representations of simple and finitely subdirectly irreducible algebras.

In [3] Hájek introduced an S5-modal expansion of any axiomatic extension $\mathcal{C}$ of his Basic Logic which is equivalent to the one-variable monadic fragment of the first-order extension $\mathcal{C}\forall$ of $\mathcal{C}$. We present next a slight generalization of his definition. Let $Prop$ be a countably infinite set of propositional variables, and let Fm be the set of formulas built from $Prop$ in the language of Basic Logic expanded with two unary connectives $\Box$ and $\Diamond$. Consider a class $\mathbb{C}$ of totally ordered BL-algebras. To interpret the formulas in Fm, consider triples $\mathbf{K} := \langle X, e, \mathbf{A}\rangle$ where X is a non-empty set, $\mathbf{A} \in \mathbb{C}$, and $e\colon X \times Prop \to A$ is a function. The *truth value* $\|\varphi\|_{\mathbf{K},x}$ of a formula φ in $\mathbf{K}$ at a point $x \in X$ is defined by recursion. For propositional variables $p \in Prop$ put $\|p\|_{\mathbf{K},x} := e(x,p)$. The definition of the truth value is then extended for the logical connectives in the language of Basic Logic in the usual way, and for the new unary connectives by

$$\|\Box\psi\|_{\mathbf{K},x} := \inf_{x'\in X} \|\psi\|_{\mathbf{K},x'}, \quad \text{and} \quad \|\Diamond\psi\|_{\mathbf{K},x} := \sup_{x'\in X} \|\psi\|_{\mathbf{K},x'}.$$

Note that the infima and suprema above may not exist in general in $\mathbf{A}$; hence, we restrict our attention to *safe* structures, that is, structures $\mathbf{K}$ for which $\|\varphi\|_{\mathbf{K},x}$ is defined for every $\varphi \in Fm$ at every point x. Given $\Gamma \subseteq Fm$, we say that a safe structure $\mathbf{K}$ is a *model* of Γ if $\|\varphi\|_{\mathbf{K},x} = 1$ for every $x \in X$ and $\varphi \in \Gamma$. For a set of formulas $\Gamma \cup \{\varphi\}$ we write $\Gamma \vDash_{\mathrm{S5}(\mathbb{C})} \varphi$ if every model of Γ is also a model of φ. The logic just defined depends on the class $\mathbb{C}$ and is denoted by $\mathrm{S5}(\mathbb{C})$. In case $\mathbb{C}$ is the class of totally ordered $\mathcal{C}$-algebras corresponding to an axiomatic extension $\mathcal{C}$ of Basic Logic we get the original definition given by Hájek; this logic was denoted by $\mathrm{S5}(\mathcal{C})$ in [3], but we reserve this notation for a related logic defined by means of an axiomatic system (see below).

We are interested in expansions of the infinite-valued Łukasiewicz logic, which we denote by $\mathcal{L}$. Recall that the equivalent algebraic semantics of $\mathcal{L}$ is the variety $\mathbb{MV}$ of MV-algebras. We write $\mathbb{MV}_{\mathrm{to}}$ for the class of totally ordered MV-algebras. Thus, $\mathrm{S5}(\mathbb{MV}_{\mathrm{to}})$ is the S5-modal expansion of $\mathcal{L}$ defined by Hájek. Consider now the logic $\mathrm{S5}(\mathcal{L})$ on the same language as $\mathrm{S5}(\mathbb{MV}_{\mathrm{to}})$ defined by the following axiomatic system:

- Axioms:

  Instantiations of axiom-schemata of $\mathcal{L}$

  $\Box\varphi \to \varphi$

[*] Speaker.

$$\varphi \to \Diamond\varphi$$
$$\Box(\nu \to \varphi) \to (\nu \to \Box\varphi)$$
$$\Box(\varphi \to \nu) \to (\Diamond\varphi \to \nu)$$
$$\Box(\varphi \vee \nu) \to (\Box\varphi \vee \nu)$$
$$\Diamond(\varphi * \varphi) \equiv (\Diamond\varphi) * (\Diamond\varphi)$$

where φ is any formula, ν is any propositional combination of formulas beginning with $\Box$ or $\Diamond$, and $\alpha \equiv \beta$ abbreviates $(\alpha \to \beta) \wedge (\beta \to \alpha)$.

- Rules of inference:

Modus Ponens: $\dfrac{\varphi, \varphi \to \psi}{\psi}$

Necessitation: $\dfrac{\varphi}{\Box\varphi}$

In [1] the authors show a strong completeness theorem stating that $\mathrm{S5}(\mathbb{MV}_{\mathrm{to}}) = \mathrm{S5}(\mathcal{L})$. In this article we study the logic $\mathrm{S5}([0,1]_{\text{Ł}})$ where $[0,1]_{\text{Ł}}$ is the standard Łukasiewicz t-norm on the unit real interval, of course $\mathrm{S5}([0,1]_{\text{Ł}})$ is a shorthand for $\mathrm{S5}(\{[0,1]_{\text{Ł}}\})$. Note that $\mathrm{S5}([0,1]_{\text{Ł}})$ is not finitary since it is a conservative expansion of the logic of $[0,1]_{\text{Ł}}$, which is not finitary. Thus, a strong completeness theorem for $\mathrm{S5}(\mathcal{L})$ with respect to $\mathrm{S5}([0,1]_{\text{Ł}})$ is not possible.

However, adding one infinitary rule to the axiomatic system defining $\mathrm{S5}(\mathcal{L})$ is enough to obtain a logic $(\mathrm{S5}(\mathcal{L})_\infty)$ strongly complete with respect to $\mathrm{S5}([0,1]_{\text{Ł}})$. This had already been shown for the propositional and first-order cases in [5]. We follow the ideas in [4] and provide an adequate algebraic representation for simple algebras needed to obtain the monadic completeness theorem. The infinitary rule in question is:

$$\frac{\Box\phi \vee (\Box\alpha \to (\Box\beta)^n) \text{ for every } n \in \mathbb{N}}{\Box\phi \vee (\Box\alpha \to \Box\alpha * \Box\beta)}.$$

We use algebraic methods to prove the completeness results stated in the previous paragraphs. The representation theorems and properties that we prove here for monadic MV-algebras are also interesting in their own right since they improve our understanding of these structures.

References

[1] Diego Castaño, Cecilia Cimadamore, José Patricio Díaz Varela, and Laura Rueda. Completeness for monadic fuzzy logic via functional algebras. *Fuzzy Sets and Systems*, 407-161-174, 2021.

[2] Diego Castaño, José Patricio Díaz Varela, and Gabriel Savoy. Strong standard completeness theorems for S5-modal Łukasiewicz logics. Submitted.

[3] Petr Hájek, On fuzzy modal logics S5(C), *Fuzzy Sets and Systems*, Volume 161, Issue 18, 2010, Pages 2389-239.

[4] Agnieszka Kulacka, Strong standard completeness for continuos t-norms. *Fuzzy Sets and Systems*, 345:139-150, 2018.

[5] F. Montagna. *Notes on Strong Completeness in Łukasiewicz, Product and BL-Logics and in Their First-Order Extensions*, volume 4460 of *Lecture Notes in Computer Science*. Springer, Berlin, Heidelberg. 2007.

[6] J. D. Rutledge, A preliminary investigation of the intinitely-many-valued predicate calculus, Ph.D. Thesis, Cornell University, 1959, 112 pp.