Many-valued coalgebraic modal logic with a semi-primal algebra of truth-degrees

Alexander Kurz[1], Wolfgang Poiger[2*], and Bruno Teheux[2]

[1] Chapman University, Orange County, California
akurz@chapman.edu

[2] University of Luxembourg, Esch-sur-Alzette, Luxembourg
wolfgang.poiger@uni.lu
bruno.teheux@uni.lu

Abstract

In this talk we report on our work in progress on a general coalgebraic approach to many-valued modal logic with a semi-primal bounded lattice expansion of truth-values. In particular, we illustrate how it relates to classical modal logic and discuss how our approach generalizes various many-valued modal logics from the literature.

1 Setting

In his generalized 'Boolean' theory of universal algebras [4] Foster introduced primal algebras. Generalizing the functional completeness of the two-element Boolean algebra $\mathbf{2}$, an algebra $\mathbf{L}$ is *primal* if every operation on its carrier set L is term-definable. During the second half of the 20th century, various weakenings of this property have been studied [9]. Since the algebras thus arising are still 'close to $\mathbf{2}$', it is reasonable to consider them as algebras of truth-values for many-valued logic. In the talk we focus on *semi-primality* [5].

Definition 1. A finite algebra $\mathbf{L}$ is *semi-primal* if every operation $f\colon L^n \to L$ which preserves subalgebras[1] is term-definable in $\mathbf{L}$.

In a slogan, semi-primal algebras are like primal algebras which allow proper subalgebras. Prominent examples from logic are finite Łukasiewicz chains or finite Łukasiewicz-Moisil chains. Further examples of semi-primal (or, more generally, quasi-primal) algebras which are not based on chains can be found among the FL_{ew}-algebras or among the *pseudo-logics*, that is, bounded lattices with an additional unary operator swapping 0 and 1. The framework of our talk is the following.

Assumption 2. Let $\mathbf{L}$ be a semi-primal algebra with underlying bounded lattice $\mathbf{L}^\flat = (L, \wedge, \vee, 0, 1)$ where $0 \neq 1$. Let $\mathcal{A} = \mathbb{HSP}(\mathbf{L})$ be the variety generated by $\mathbf{L}$.

Abstractly, $\mathbf{2}$-valued coalgebraic modal logic for an endofunctor $\mathsf{T}\colon \mathsf{Set} \to \mathsf{Set}$ is summarized in the following picture based on Stone duality after 'forgetting topology':

$$\mathsf{T} \circlearrowright \mathsf{Set} \rightleftarrows \mathsf{BA} \circlearrowleft \mathsf{A} \qquad (1)$$

For example, if $\mathsf{T} = \mathcal{P}$ is the covariant powerset functor, then the category of $\mathcal{P}$-coalgebras $\mathsf{Coalg}(\mathcal{P})$ corresponds to the category Kripke frames with bounded morphisms. Similarly the category of A-algebras $\mathsf{Alg}(\mathsf{A})$ corresponds to the variety of modal algebras.

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*Speaker

[1] If $\mathbf{S}$ is a subalgebra of $\mathbf{L}$ then $a_1 \ldots a_n \in S \Rightarrow f(a_1, \cdots a_n) \in S$.

To get a similar picture for our variety $\mathcal{A}$, we apply the duality for semi-primal varieties due to Keimel and Werner [7] (also see [3]) which asserts that $\mathcal{A}$ is dually equivalent to the category $\mathsf{Stone}_{\mathbf{L}}$ defined as follows

Definition 3. Objects of $\mathsf{Stone}_{\mathbf{L}}$ are of the form $(X, \mathbf{v})$ where $X \in \mathsf{Stone}$ and $\mathbf{v}\colon X \to \mathbb{S}(\mathbf{L})$ is continuous. Morphisms $f\colon (X, \mathbf{v}) \to (Y, \mathbf{w})$ in $\mathsf{Stone}_{\mathbf{L}}$ are continuous maps satisfying $\mathbf{w}(f(x)) \leq \mathbf{v}(x)$.

Let $\mathsf{Set}_{\mathbf{L}}$ be the category obtained from $\mathsf{Stone}_{\mathbf{L}}$ after 'forgetting topology'. There is a canonical way to lift T from diagram (1) to an endofunctor $\mathsf{T}'\colon \mathsf{Set}_{\mathbf{L}} \to \mathsf{Set}_{\mathbf{L}}$. We ultimately aim to describe the modal logic abstractly characterized by

$$\mathsf{T}' \circlearrowright \mathsf{Set}_{\mathbf{L}} \rightleftarrows \mathcal{A} \circlearrowleft \mathsf{A}' \tag{2}$$

This also yields the more commonly investigated case

$$\mathsf{T} \circlearrowright \mathsf{Set} \rightleftarrows \mathcal{A} \circlearrowleft \mathsf{A}' \tag{3}$$

obtained after composing by the forgetful functor $\mathsf{U}\colon \mathsf{Set}_{\mathbf{L}} \to \mathsf{Set}$ and its left adjoint.

2 Examples

Example 4. In our first example, let $\mathsf{T} = \mathcal{P}$. The coalgebras for the lifted functor $\mathsf{Coalg}(\mathcal{P}')$ correspond to *crisp* $\mathbf{L}$*-frames*. That is, to triples $\mathfrak{F} = (W, R, \mathbf{v})$ where (W, R) is a Kripke frame and $\mathbf{v}\colon W \to \mathbb{S}(\mathbf{L})$ satisfies the compatibility condition

$$wRw' \Rightarrow \mathbf{v}(w') \subseteq \mathbf{v}(w)$$

For the $\mathbf{L}$*-models* over $\mathfrak{F}$ we only allow valuations $Val : W \times \mathsf{Prop} \to L$ which always satisfy

$$Val(w, p) \in \mathbf{v}(w).$$

In this case, diagram (2) is closely related to work by Maruyama [8]: the algebras $\mathsf{Alg}(\mathsf{A}')$ correspond to what is therein called $\mathbb{ISP}_{\mathbb{M}}(\mathbf{L})$. The non-restricted case where all valuations are allowed corresponds to diagram (3) and arises if $\mathbf{v}(w) = \mathbf{L}$ everywhere. Here, in the special case $\mathbf{L} = \mathbf{Ł}_n$ it corresponds to modal extensions of Łukasiewicz many-valued logic as described in [6].

Example 5. For another example, we hint at the case where $\mathsf{T} = \mathcal{L}$ is the covariant functor which generalizes $\mathcal{P}$, that is, it is defined on objects by $\mathcal{L}(X) = L^X$ and on morphisms $f : X \to Y$ by

$$\begin{aligned} \mathcal{L}f : L^X &\to L^Y \\ h &\mapsto (y \mapsto \bigvee\{h(x) \mid f(x) = y\}). \end{aligned}$$

Now in (2) the coalgebras for the lifted endofunctor $\mathsf{Coalg}(\mathcal{L}')$ correspond to the $\mathbf{L}$-*labeled* $\mathbf{L}$-*frames*, that is, $(W, R, \mathbf{v})$ similar to the crisp $\mathbf{L}$-frames except that now the accessibility relation

$$R\colon W \to L^W$$

is many-valued as well. Diagram (3) corresponds again to $\mathbf{L}$-labeled frames without further restrictions. This, in the case $\mathbf{L} = \mathbf{Ł}_n$ corresponds to the frames that have been recently investigated by algebraic means in [2] (see also [1]).

3 Content of the talk

In the talk, we will report about our work in progress on the investigation of the modal logics arising from diagrams (2) and (3) in the general case, and discuss some examples which arise by specifying to some particular functors T.

We explain how our logics relate to the classical case given by diagram (1). Afterwards we may discuss our results regarding completeness, definability and expressivity of these logics.

References

[1] Félix Bou, Francesc Esteva, Lluís Godo, and Ricardo Oscar Rodríguez. On the minimum many-valued modal logic over a finite residuated lattice, 2009.

[2] Cordero P. Busaniche, M. and R.O. Rodriguez. Algebraic semantics for the minimum many-valued modal logic over $\mathbf{Ł}_n$., 2022.

[3] David M. Clark and Brian A. Davey. Natural dualities for the working algebraist. cambridge studies in advanced mathematics, 1998.

[4] Alfred L. Foster. Generalized "boolean" theory of universal algebras. part i., 1953.

[5] Alfred L. Foster and Alden Pixley. Semi-categorical algebras. i. semi-primal algebras., 1964.

[6] G. Hansoul and B. Teheux. Extending Łukasiewicz logics with a modality: Algebraic approach to relational semantics., 2013.

[7] Klaus Keimel and Heinrich Werner. Stone duality for varieties generated by quasi-primal algebras, 1974.

[8] Yoshihiro Maruyama. Natural duality, modality, and coalgebra, 2012.

[9] Robert W. Quackenbush. Appendix 5: Primality: the influence of boolean algebras in universal algebra, 1979.