From contact relations to modal opetators, and back

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One of the standard axioms for Boolean contact algebras says that if a region x is in contact with the join of y and z, then x is in contact with at least one of the two regions. Our intention is to examine a stronger version of this axiom according to which if x is in contact with the supremum of some family S of regions, then there is y in S that is in contact with x.

Any Boolean algebra is turned into a *Boolean contact algebra* by expanding it to a structure $\mathfrak{B} = \langle B, \cdot, +, -, \mathbf{0}, \mathbf{1}, \mathsf{C}\rangle$ where $\mathsf{C} \subseteq B^2$ is a *contact* relation which satisfies the following five axioms:

$$\neg(\mathbf{0}\,\mathsf{C}\,x), \tag{C0}$$

$$x \leq y \wedge x \neq \mathbf{0} \longrightarrow x\;\mathsf{C}\;y, \tag{C1}$$

$$x\;\mathsf{C}\;y \longrightarrow y\;\mathsf{C}\;x, \tag{C2}$$

$$x \leq y \wedge z\;\mathsf{C}\;x \longrightarrow z\;\mathsf{C}\;y\,, \tag{C3}$$

$$x\,\mathsf{C}\,y + z \longrightarrow x\,\mathsf{C}\,y \vee x\,\mathsf{C}\,z\,. \tag{C4}$$

In this study, we consider *complete* Boolean contact algebras in which the contact *completely* distributes over join, i.e., those that satisfy the following second-order constraint:

$$x\,\mathsf{C}\bigvee_{i\in I} x_i \longrightarrow (\exists i \in I)\,x\,\mathsf{C}\,x_i\,. \tag{C4$^{\mathrm{c}}$}$$

It is clear that (C4$^{\mathrm{c}}$) entails (C4), yet the converse implication is not true in general. The main objective of the talk is to present the consequences of adopting (C4$^{\mathrm{c}}$) as an axiom, provide several examples, and analyze a modal possibility operator that is definable in the class of contact algebras satisfying the aforementioned stronger version of (C4).

1 A modal operator

It is known that the relation of *subordination* on a Boolean algebra is a natural generalization of the notion of modal operator. For instance, modal operators give rise to special subordinations called by [1] *modally definable*. Moreover, the authors prove that modal operators on a Boolean algebra are in one-to-one correspondence with modally definable subordinations. Contact relations give rise to non-tangential inclusion that is a special case of the subordination relation:

$$x \ll y \;:\longleftrightarrow\; x \not\mathsf{C} -y\,,^{1} \tag{df $\ll$}$$

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and in the talk we show that there is a correspondence between contact and modal operators. The crucial observation is that if $\mathfrak{B}$ is a complete Boolean contact algebra, then $\mathfrak{B}$ satisfies (C4^{c}) iff for every region x there exists a unique region y such that $\mathsf{C}(x) = \mathsf{O}(y)$, where:

$$\mathsf{C}(x) \coloneqq \{y \in B \mid x \,\mathsf{C}\, y\}\,,$$
$$\mathsf{O}(x) \coloneqq \{y \in B \mid x \cdot y \neq \mathbf{0}\}\,.$$

The uniqueness property entails existence of an operation $m\colon B \longrightarrow B$ such that:

$$m(x) \coloneqq (\iota y)\ \mathsf{C}(x) = \mathsf{O}(y)\,.^{2} \qquad (\mathtt{df}\ m)$$

We prove that m is a modal possibility operator. Obviously, we have that:

$$x \,\mathsf{C}\, y \longleftrightarrow m(x) \cdot y \neq \mathbf{0}\,,$$

and so:

$$x \ll y \longleftrightarrow m(x) \leq y\,.$$

Recall that any modal algebra $\mathfrak{B} \coloneqq \langle B, \Diamond\rangle$ whose possibility operator satisfies the following two conditions:

$$x \leqslant \Diamond x\,, \qquad (\mathrm{T}_\Diamond)$$
$$\Diamond\Box x \leq x\,, \qquad (\mathrm{B}_\Diamond)$$

where $\Box \coloneqq -\Diamond-$, is a *KTB-algebra.*

If $\mathfrak{B}$ is a complete Boolean contact algebra that satisfies (C4^{c}), then $m\colon B \longrightarrow B$ is a completely additive modal possibility operator such that $\langle B, m\rangle$ is a KTB-algebra. So, under our assumptions the non-tangential inclusion is a modally definable subordination.

On the other hand, if $\mathfrak{B}$ is a complete KTB-algebra, then:

$$\mathsf{C}_\Diamond \coloneqq \{\langle x, y\rangle \mid x \cdot \Diamond y \neq \mathbf{0}\} \qquad (\mathtt{df}\ \mathsf{C}_\Diamond)$$

is a contact relation that satisfies (C4^{c}). Moreover, $\Diamond = m$, where m is the modal operator for $\mathsf{C}_\Diamond$ introduced by ($\mathtt{df}\ m$).

2 The isomorphism of categories

Let $\mathbf{C4^c}$ be the class of complete Boolean contact algebras that satisfy (C4^{c}). We endow this class with certain morphisms in order to obtain a category. Given two algebras $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathbf{C4^c}$, a mapping $h\colon B_1 \to B_2$ is a *p-morphism*[3] iff it is a homomorphism such that:

$$h(x)\,\mathsf{C}_2\,h(y) \longrightarrow x \,\mathsf{C}_1\, y\,, \qquad (\mathrm{P1})$$
$$h(z) \ll_2 y \longrightarrow (\exists x \in B_1)(z \ll_1 x \wedge h(x) \leq_2 y)\,. \qquad (\mathrm{P2})$$

Of course, (P1) is equivalent to:

$$x \ll_1 y \longrightarrow h(x) \ll_2 h(y)\,.$$

[1] $-y$ is the Boolean complement of y.
[2] ι is the uniqueness operator, i.e., $(\iota x)\,\varphi(x)$ denotes the only object x that satisfies $\varphi(x)$.
[3] The idea of this comes from [2], where similar morphisms are called q-morphism.

The class $\mathsf{C4^c}$ together with p-morphisms form a category with the identity functions serving as the identity morphisms.

Also, let $\mathsf{KTB^c}$ be the class of all complete KTB algebras, that we turn into a category by taking as morphisms the standard Boolean homomorphisms preserving the possibility operator.

We show that there is a covariant functor $F\colon \mathsf{C4^c} \to \mathsf{KTB^c}$ which sends a complete BCA satisfying (C4^c) to a complete modal algebra, and such that for every $f \in \mathrm{Hom}_{\mathsf{C4^c}}(B_1, B_2)$, f is also an arrow in $\mathrm{Hom}_{\mathsf{KTB^c}}(B_1, B_2)$, i.e. $F(f) = f$. Analogously, there is a covariant functor $G\colon \mathsf{KTB^c} \to \mathsf{C4^c}$, which takes every $h \in \mathrm{Hom}_{\mathsf{KTB^c}}(B_1, B_2)$ to h itself.

Moreover, it is the case that:

$$G \circ F = 1_{\mathsf{C4^c}} \qquad \text{and} \qquad F \circ G = 1_{\mathsf{KTB^c}}\,,$$

where $1_{\mathsf{C4^c}}$ and $1_{\mathsf{KTB^c}}$ are the identity functors for the respective categories. In consequence we obtain that the categories $\mathsf{C4^c}$ and $\mathsf{KTB^c}$ are isomorphic.

3 Resolution contact algebras

Resolution contact algebras form a proper subclass of $\mathsf{C4^c}$ and serve as a spatial interpretation of both the contact relation that satisfies (C4^c) and the modal operator defined via the contact. The inspiration for this comes from [5], [6] and [3].

A *partition* of a Boolean algebra $\mathfrak{B}$ is any non-empty set P of non-zero and disjoint regions of B that add up to the unity: $\bigvee P = \mathbf{1}$. Let $\mathfrak{B}$ be a complete Boolean contact algebra, let $P \coloneqq \{p_i \mid i \in I\}$ be its partition. We define:

$$x \mathrel{\mathsf{C}_P} y \;:\longleftrightarrow\; (\exists i \in I)\,(x \mathrel{\mathsf{O}} p_i \wedge y \mathrel{\mathsf{O}} p_i)\,. \tag{df C_P}$$

C_P is a contact relation which satisfies (C4^c). For every element p_i of the partition, $\langle {\downarrow} p_i, \mathsf{C}_i\rangle$ where $\mathsf{C}_i \coloneqq \mathsf{C}_P \cap ({\downarrow} p_i \times {\downarrow} p_i)$ is a Boolean contact algebra with the full contact relation, so in particular, it satisfies (C4^c).

We adopt the following conventions: every partition of $\mathfrak{B}$ will be called its *resolution*[4], and the elements of the partition will be called *cells*. Any Boolean algebra expanded with C_P for a given partition P will be called *resolution contact algebra*. RCA is the class of such algebras, and $\mathsf{RCA^c}$ is its subclass composed of complete resolution algebras. In the case $x \mathrel{\mathsf{C}_P} y$ we will say that x is in *c-contact* with y.

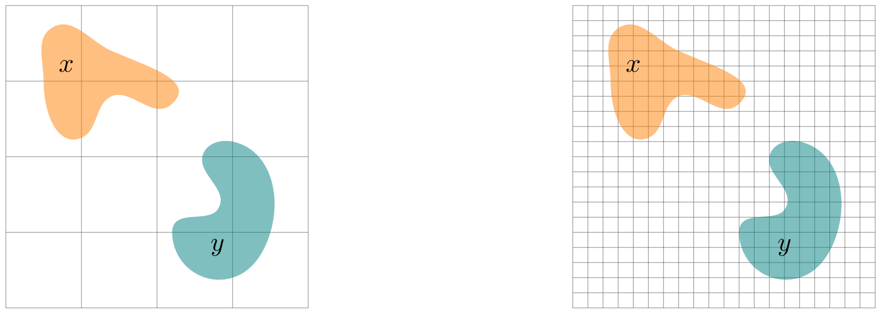
The fineness of the partition is a counterpart of the precision with which we can discern regions and their mutual relations. For example, the regions x and y in Figure 1a are in c-contact, since they overlap a common cell from the sixteen element partition. From the perspective of the picture those regions may seem to be way apart, but we can think of the resolution as the frame of reference for comparison of regions with respect to C_P relation. The finer the resolution, the more precise approximation of contact between regions, as we can see in Figure 1b.

In every resolution algebra: $m(m(x)) = m(x)$, so m is a closure operator in every such algebra.

If $\mathfrak{B} \in \mathsf{RCA^c}$ has a finite resolution $P = \{p_i \mid i \leqslant n\}$ for some $n \in \mathbb{N}$, then the Kripke relation on the set $\mathrm{Ult}\,B$ is an equivalence relation and there is a one-to-one correspondence $f\colon P \to \mathrm{Ult}\,B/_R$ between cells and equivalence classes of ultrafilters.

Moreover, the contact determined by partition is related to S5 modal operators in the sense of the following:

[4]The name comes from [6], yet unlike there we do not limit it to finite partitions.



(a) (b)

Figure 1: On the left: regions x and y that are in contact with respect to a partition consisting of sixteen cells. On the right: regions x and y are no longer in contact if we take a finer partition as the frame of reference.

Theorem 3.1. *Given an S5 modal algebra* $\mathfrak{B} = \langle B, \Diamond \rangle$*, its expansion* $\mathfrak{B}^{\star} = \langle B, \Diamond, \mathsf{C}_{\Diamond} \rangle$ *can be embbeded into a modal expansion of a resolution algebra.*

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