Proof Theory for Intuitionistic Temporal Logic

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Topological dynamics is a branch of dynamical systems theory which studies the asymptotic behaviour of continuous functions on topological spaces. A (topological) dynamic system is a topological space $\mathcal{X} = (X, \tau)$ equipped with a continuous function $S : X \longrightarrow X$. Based on Tarski's observations that modal logic can be evaluated in topological spaces [10], Artemov et al. introduced in 1997 a temporal logic which extends modal logic by the next operator $\bigcirc$ to reason about topological dynamic systems [1]. From a temporal point of view the continuous function S can be regarded as a time-function which maps points of the topological space from one time moment to the next. The next operator is therefore used to reason about the behaviour of S. The work of Artemov et al. was later continued by Kremer and Mints [6] by extending their system with the temporal operators $\Diamond$ called eventually and $\Box$ called henceforth. The resulting system is called Dynamic Topological Logic (DTL). The addition of $\Diamond$ and $\Box$ substantially increases the expressive power of DTL and allows one to formulate interesting properties of dynamical systems. The project to build a logic to reason about topological dynamics however suffered a setback when Konev et al. proved that DTL is not decidable [5]. As a consequence of this result the focus of the project has shifted from DTL to an intuitionistic variant of DTL called Intuitionistic Temporal Logic (ITL). This focus shift is motivated by the observation that intuitionistic logic has better computational properties than classical logic and so it is hoped that ITL is decidable. Indeed, first results about ITL are promising: In 2018, Fernández-Duque established decidability of a fragment of ITL called $\mathsf{ITL}_\Diamond$ which only contains the next and the eventually operator [4]. Importantly, henceforth and eventually are not interdefinable in ITL (in contrast to DTL) as the base logic of ITL is intuitionistic. The proof of decidability relies on model theoretic techniques, in particular on the construction of so-called quasi models. Later, Boudou et al. proved completeness of this fragment with respect to the class of topological dynamic systems [2] by using similar techniques.

While the semantical aspects of ITL have been studied quite extensively in recent years, there is little known about the proof theory of ITL. Our long term goal is to fill this gap and provide a satisfying proof theory for intuitionistic temporal logic. For a start, we aim to investigate the proof theory of $\mathsf{ITL}_\Diamond$. Our project roughly consists of three main steps:

1. Define a sound and complete cyclic proof system for $\mathsf{ITL}_\Diamond$.
2. Establish cut-elimination either syntactically or by an indirect argument.
3. Use the cut-free system to obtain a syntactic decidability proof and invesitigate the complexity of the validity problem.

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At the point of writing this abstract we have completed step 1 and we are currently investigating the second step. In the following we describe in more detail each step.

For step 1 we define a cyclic proof system called $\mathsf{ITL}^c_\Diamond$ which is based on a standard multi-conclusion sequent calculus for intuitionistic logic presented in [7]. This calculus is extended by rules for the next operator and the eventually operator. In particular, the rules for $\Diamond$ are standard unfolding rules, which replace the formula $\Diamond A$ by its equivalent unfolding $A \vee \bigcirc\Diamond A$. The rules for $\Diamond$ together with the cycle mechanism characterize the formula $\Diamond A$ as the least fixed point of the function $X \mapsto A \vee \bigcirc X$. Importantly, as the logic $\mathsf{ITL}_\Diamond$ is not only sound and complete with respect to topological dynamic models but also with respect to the class of dynamic Kripke frames [2], the topological semantics does not play a role in the presented calculus. As henceforth is not definable in our language, there does not exist any form of fixed point alternation in $\mathsf{ITL}_\Diamond$. This implies that characterizing successful repetitions in a cyclic proof is a much easier task than for other fixed point logics such as the modal mu-calculus. In particular, we do not require a focus mechanism for our system. Soundness of $\mathsf{ITL}^c_\Diamond$ is established by a minimal counter model approach which is common in the literature (see for example [9]). For completeness we consider a Hilbert style proof system for $\mathsf{ITL}_\Diamond$ which is proven to be complete with respect to the class of topological dynamic systems in [2] and show how to embed it into the cyclic calculus $\mathsf{ITL}^c_\Diamond$. As a consequence of this technique we do not obtain cut-free completeness, as the cut-rule is needed to derive the modus ponens rule. An important goal of our work is therefore to also establish cut free completeness, which brings us to step 2.

For step 2 we plan to establish a cut-elimination result by providing a syntactic cut-elimination procedure inspired by the continuous cut-elimination procedure of Savateev and Shamkanov in [8]. To that end we define a non-wellfounded proof system called $\mathsf{ITL}^n_\Diamond$ for $\mathsf{ITL}_\Diamond$. We first show how to unfold a cyclic proof into a non-wellfounded proof and vice versa, how to prune a non-wellfounded proof into a cyclic one. By doing so we establish soundness and completeness of the non-wellfounded system. Then a procedure is described which, given a non-wellfounded proof in which the cut-rule is applied, pushes the occurrence of the cut-rule upwards. By applying the procedure infinitely many times, we create an infinite sequence of non-wellfounded proofs which has the property that in each proof the first appearence of cut occurs above the first appearence of cut in the previous proof. By taking the limit of this construction, we obtain a cut-free non-wellfounded proof which can be pruned back into a cyclic proof. The main difficulty in this approach lies in showing that the limit of this sequence is a tree which satisfies the global trace conditions required for soundness of the system. In case such a cut-elimination procedure does not work for $\mathsf{ITL}^c_\Diamond$ we would consider establishing cut-elimination indirectly by giving a completeness proof without cut via a standard proof search argument.

Finally, for step 3, we plan to establish decidabilty of $\mathsf{ITL}_\Diamond$ by translating the non-wellfounded calculus $\mathsf{ITL}^n_\Diamond$ minus cut into a parity game called proof search game. This proof search game is played by two players called Prover and Refuter. It is Prover's goal to show that a given sequent is derivable in $\mathsf{ITL}^n_\Diamond$ and Refuter's goal to show the opposite, i.e. the sequent is not derivable. The positions of the game include all sequents that can be built from formulas in the Fischer-Ladner closure of the given sequent as well as every possible rule application including only such sequents. Whenever a match is in a position which is a sequent, it is Prover's turn and she can choose which rule to apply to that sequent. Next, it is Refuter's turn who can choose at which premise of the rule instance chosen by Prover the game continues. Therefore, a match in the proof search game corresponds to a finite or infinite path of a $\mathsf{ITL}^n_\Diamond$-pre-proof.

Consequently, the strategy tree of Prover corresponds to some $\mathsf{ITL}^n_\diamond$-pre-proof. Observe that this corresponding pre-proof is analytic, as the game only consists of sequents that occur in the Fischer-Ladner closure of the endsequent. We establish the result that a sequent is provable in $\mathsf{ITL}^n_\diamond$ if and only if Prover has a positional winning strategy in the corresponding game. We then use a result proven by Calude et al. in [3] to establish the existence of an algorithm deciding for each sequent whether Prover has a positional winning strategy in the corresponding game and so whether the sequent is provable in $\mathsf{ITL}^n_\diamond$. The aforementioned result also establishes a first complexity bound for the validity problem of $\mathsf{ITL}_\diamond$. However, it is unclear whether such an approach would give us an optimal complexity bound. We plan to investigate this question and to give alternative decision procedures.

Our work is a continuation of the project to develop logics for reasoning about topological dynamics with good computational properties. We hope to provide a first insight into the proof theory of intuitionistic temporal logics and lay a foundation to investigate more complicated logics, in particular the logic ITL based on the full language with next, eventually and henceforth. The work on cut elimination is especially interesting, as surprisingly little can be found about this topic for cyclic proofs in general and we are interested in filling this gap. Furthermore, we hope to provide a new proof of decidability of $\mathsf{ITL}_\diamond$ which, in contrast to [4], relies entirely on syntactic arguments.

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