# Free algebras in all subvarieties of the variety generated by the MG t-norm

Noemí Lubomirsky[1,*] and José Patricio Díaz Varela[2]

[1] CMaLP-CIC-UNLP and CONICET
nlubomirsky@mate.unlp.edu.ar

[2] Departamento de Matemática (UNS) and INMABB (CONICET)
usdiavar@criba.edu.ar

Basic Fuzzy Logic (BL for short) was introduced by Hájek in [7] to formalize fuzzy logics in which the conjunction is interpreted by a continuous t-norm on the real segment $[0,1]$ and the implication by its corresponding adjoint. The equivalent algebraic semantics for BL, in the sense of Blok and Pigozzi, is the variety of BL-algebras $\mathcal{BL}$. Many algebraic properties of BL-algebras correspond to logical properties of BL. For example, and what is our concern, the elements of free algebras in $\mathcal{BL}$ are in correspondence with equivalence classes of formulas in the logic. This is why many attempts to study free BL-algebras have been accomplished in the last decades. Some of these studies, like [6] and [4], describe free algebras in subvarieties of BL-algebras from an structural point of view, considering the representation of the algebra as weak boolean product of directly indecomposable BL-algebras over the Stone Space corresponding to a free Boolean algebra. Some others, as [9] and [2] present a functional description of the elements of the free algebra.

In [5] there is a functional representation of the free algebra in the subvariety of BL-algebras generated by a chain which is the ordinal sum of the standard MV-algebra $[0,1]_{\mathbf{MV}}$ and a basic hoop $\mathbf{H}$, that is, generated by $[0,1]_{\mathbf{MV}} \oplus \mathbf{H}$. The main advantage of this approach, is that unlike the work done in [3] and [2], when the number $n$ of generators of the free algebra increase the generating chain remains fixed. This provides a clear insight of the role of the two main blocks of the generating chain in the description of the functions in the free algebra: the role of the regular elements and the role of the dense elements.

We will focus on the particular case of the variety $\mathcal{MG}$, the variety generated by the ordinal sum of the standard algebra $[0,1]_{\mathbf{MV}}$ and the Gödel hoop $[0,1]_{\mathbf{G}}$. As a logical counterpart, this variety $\mathcal{MG}$ has also an equational characterization as a subvariety of $\mathcal{BL}$, given by adding the equation

$$(\neg\neg x \to x)^2 = \neg\neg x \to x$$

to the axioms of Basic Logic. This equation show that in our variety, the dense elements are idempotent. The t-norm that generates this variety (which we call MG t-norm) is the function $t : [0,1]^2 \to [0,1]$ defined by

$$t(x,y) = \begin{cases} max\{0, x+y-\frac{1}{2}\} & \text{if } x,y \in [0,\frac{1}{2});\\ min\{x,y\} & \text{otherwise.}\end{cases}$$

The functions in the free algebra $Free_{\mathcal{MG}}(n)$ can be described by decomposing the domain of the functions $([0,1]_{\mathbf{MV}} \oplus [0,1]_{\mathbf{G}})^n$ in a finite number of pieces. On each piece a function $F \in Free_{\mathcal{MG}}(n)$ coincides either with a function in $Free_{\mathcal{MV}}(n)$ or a function in $Free_{\mathcal{G}}(m)$, for some $m \leq n$.

*Speaker.

It is well known that there exists a one-to-one correspondence between subvarieties of BL-algebras and axiomatic extensions of the BL-logic, through a natural translation between algebraic equations and logical axioms. Therefore, when we study a family of subvarieties of $\mathcal{BL}$ and their equational bases, we also obtain the corresponding axiomatic extensions of BL. In [1] there is a description of the lattice of subvarieties of $\mathcal{BL}$, but in our case we will completely describe the lattice of subvarieties of a particular subvariety: $\mathcal{MG}$. For this purpose, we will first characterize the join-irreducible subvarieties in the lattice and then show that every subvariety is a join of finite join-irreducible varieties in the lattice. With that results, we will then give an equational characterization for every subvariety in the lattice.

Finally, we give a characterization for every subvariety of $\mathcal{MG}$ as a finite product of algebras given by the restriction of the functions in $Free_{\mathcal{MG}}(n)$ over some rational points and their neighbourhoods. This result, extends the characterization given by G. Panti in [10] for all subvarieties of $\mathcal{MV}$ and the description of the algebra $Free_{\mathcal{MG}}(n)$ given in [8], since we give in this case a description of the free algebra for every subvariety of $\mathcal{MG}$.

# References


[1] Aglianò, P., Montagna, F., *Varieties of BL-Algebras II* Studia Logica, 2018, vol. 106, N. 4, pag. 721–737.

[2] Aguzzoli, S. and Bova,S. *The free n-generated BL-algebra* Annals of Pure and Applied Logic, 2010, N. 161, pag. 1144–1170.

[3] Bova, S. *BL-functions and Free BL-algebra*, PhD thesis, University of Siena.

[4] Busaniche, M., *Free algebras in varieties of BL-algebras generated by a chain*, Algebra univers., 2003, N. 50, pag. 259–277.

[5] Busaniche, M., Castiglioni, J.L. and Lubomirsky, N., *Functional representation of finitely generated free algebras in subvarieties of BL-algebras*, Annals of Pure and Applied Logic, 2020, N. 171, pag. 1–22.

[6] Cignoli, R., Torrens, A., *Free algebras in varieties of BL-algebras with a Boolean retract*, Algebra Universalis, 2002, N. 48, pag. 55–79.

[7] Hájek, P. *Metamathematics of Fuzzy Logic*, Kluwer, 1998.

[8] Lubomirsky, N., *Técnicas geométricas y combinatorias en el estudio de subvariedades de BL-álgebras*, PhD. Thesis, National University of La Plata, 2017.

[9] Montagna, F., *The Free BL-algebra on One Generator*, Neural Network World, 2000, Vol. 10.

[10] Panti, G., *Varieties of MV-Algebras*, Journal of Applied Non-Classical Logics, 199, Vol. 9, N 1, pag. 141–157.